EXERCISES FOR THE COURSE "PROPERTY (T), FIXED POINT PROPERTIES AND STRENGTHENING" IN WROCŁAW (OCTOBER, 2–13, 2017)

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Problems with $^{\#}$ might be more difficult than the other ones. Problems with * might be specially educationally good.

Throughout this lecture series and this exercise session, all topological groups are assumed to be Hausdorff; we equip finitely generated groups with discrete topology. For $p \in [1, \infty]$, we denote by ℓ_p the real ℓ_p-space based on an infinite countable set (so, ℓ_p is isometrically isomorphic to $\ell_p(\mathbb{N}, \mathbb{R})$).

1. Expander graphs

In this section, let $\Gamma = (V, E)$ be a finite ($|V| < \infty$ and $|E| < \infty$), non-oriented graph, where V is the set of vertices and E is the set of non-oriented edges. Denote by $\overrightarrow{E}$ the set of oriented edges. (For each non-oriented edge, we put an orientation on it, and regard the resulting oriented edge as an element in $\overrightarrow{E}$. We do this procedure for both of possible two orientations of a non-oriented edge; so, each non-oriented edge results in two oriented edges.) Set $n = n_\Gamma$ to be $|V|$ and $k = k_\Gamma$ to be the maximum degree of Γ. Denote by $\Delta = \Delta_\Gamma$ the (non-normalized) graph Laplacian. Namely,

$$\Delta = \frac{1}{2} d^* d \colon \ell_2(V, \mathbb{R}) \to \ell_2(V, \mathbb{R}),$$

where $d \colon \ell_2(V) \to \ell_2(\overrightarrow{E}, \mathbb{R})$ is the discrete gradient operator,

$$(df)(\overrightarrow{e}) = f(\overrightarrow{e}^{\,+}) - f(\overrightarrow{e}^{\,-}), \quad f \in \ell_2(V, \mathbb{R}),\ \overrightarrow{e} \in \overrightarrow{E},$$

and $d^* \colon \ell_2(\overrightarrow{E}, \mathbb{R}) \to \ell_2(V, \mathbb{R})$ is its adjoint. By $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$, we mean the enumeration of the eigenvalues of Δ with multiplicities.

Problem 1 (Matrix form of Δ)**.** Prove that $\Delta = D - A$. Here $D = D_\Gamma$ is the degree operator (matrix), i.e.,

$$D_{u,v} = \begin{cases} \deg(u) & \text{if } u = v, \\ 0 & \text{otherwise} \end{cases}$$

and $A = A_\Gamma$ is the adjacent operator (matrix), i.e., $A_{u,v}$ is the number of edges in $\overrightarrow{E}$ that are from u to v. We take the convention that a self-loop contributes to the degree twice.

(Hint: For $u, v \in V$, compute $\langle \Delta\delta_u, \delta_v\rangle_{\ell_2(V)}$.)

Problem 2. Show that $\lambda_0 = 0$. Moreover, prove that $\lambda_1 > 0$ if Γ is connected.

Problem 3. Prove that $\lambda_{n-1} \leq 2k$.

Problem 4. Let K_n be the complete graph of n vertices (each distinct pair of vertices is connected by an exactly one edge). Compute $\lambda_0, \ldots, \lambda_{n-1}$.

Problem* 5. Verify one side of the discrete Cheeger inequality:

$$\frac{1}{2}\lambda_1 \leq h.$$

Here $h = h(\Gamma)$ is the (non-normalized and vertex) isoperimetric constant, namely,

$$h(\Gamma) = \inf\left\{\frac{|\partial A|}{|A|} : A \subseteq V,\ 1 \leq |A| < \frac{|V|}{2} + 1\right\}.$$

Here ∂A means the edge boundary: $\partial A = \{\overrightarrow{e} \in \overrightarrow{E} : \overrightarrow{e}^- \in A,\ \overrightarrow{e}^+ \in V \setminus A\}$.

(Hint: Apply the variational formula for λ_1 to a certain function $f \in \ell_2(V)$.)

Problem$^\#$ 6. Prove the other side of the discrete Cheeger inequality:

$$h \leq \sqrt{2k\lambda_1}.$$

Problem 7. Show that the notion of (ordinary) expanders coincides with that of Banach $(\ell_2, 2)$-anders.

Problem 8 (Fréchet embedding)**.** Show that every Γ embeds *isometrically* into ℓ_∞^n. Here ($n = n_\Gamma$ and) ℓ_∞^n denotes the n-dimensional real ℓ_∞-space.

Problem* 9 (Lazy random walk on expanders)**.** Let $(\Gamma_m = (V_m, E_m))_{m\in\mathbb{N}}$ be an expander family such that all Γ_m are simple (no self-loops or multiple edges) and k-regular (for a fixed $k \geq 2$). Set $\epsilon > 0$ to satisfy $\epsilon \leq \lambda_1(\Gamma_m)$ for all m. Set $n_m = |V_m|$.

For each m, fix $w_m \in V_m$ (base point). Then, the *lazy random walk* on Γ_m starting at w_m is defined as follows: the random walk starts at w_m at time $t = 0$. From time l to time $l + 1 (l \in \mathbb{N})$, stay at the same vertex as at $t = l$ with probability $1/2$; with probability $1/2$, move to one of the adjacent vertices that is chosen uniformly at random (independently from the history of the walk in the past).

Set $\mu_m^{(l)}$ to be the probability distribution of the lazy random walk at time l, i.e., for $v \in V_m$, $\mu_m^{(l)}(v)$ is the probability that the lazy random walk is at v at time $t = l$. Regard $\mu_m^{(l)}$ as a (norm 1) vector in $\ell_1(V_m)$, and write it as $\boldsymbol{\mu}_m^{(l)}$. Let $\boldsymbol{\nu}_m$ be the vector in $\ell_1(V_m)$ corresponding to the uniform distribution on V_m, namely, $\nu_m(v) = 1/n_m$ for all $v \in V_m$.

In this exercise, we define the *mixing time* t_m for the lazy random walk above on Γ_m in the following manner:

$$t_m = \min\left\{l \in \mathbb{N} : \|\boldsymbol{\mu}_m^{(l)} - \boldsymbol{\nu}_m\|_1 \leq \frac{1}{3n_m}\right\}.$$

Here $\|\cdot\|_1$ means the ℓ_1-norm.

(1) Show that for all m, $t_m \geq \frac{\log n_m}{\log k} - 1$.

(Hint: If $\|\boldsymbol{\mu}_m^{(l)} - \boldsymbol{\nu}_m\|_1 \leq \frac{1}{3n_m}$, then in particular, for every $v \in V_m$, $\mu_m^{(l)}(v) > 0$.)

(2) Describe the Markov operator M_m for the lazy random walk above on Γ_m in terms of $\Delta_m = \Delta_{\Gamma_m}$. In other words, find an operator M_m acting on $\ell_1(V_m)$ such that for all $l \in \mathbb{N}$,

$$\boldsymbol{\mu}_m^{(l)} = M_m^l \boldsymbol{\mu}_m^{(0)} (= M_m^l \delta_{w_m}).$$

(3) Show that for all m,

$$t_m \leq \frac{3}{2} \cdot \frac{\log(6n_m)}{\left(-\log\left(1 - \frac{\epsilon}{2k}\right)\right)}.$$

(Hint. By Cauchy–Schwarz, observe that for every $v \in \mathbb{R}^{V_m}$,

$$\|v\|_1 \leq \sqrt{n_m} \cdot \|v\|_2$$

holds true.)

Since for $0 < \delta < 1$, $-\log(1-\delta) > \delta$ holds, we conclude that *for expander graphs, the mixing times t_m for lazy random walks on Γ_m have the order*

$$t_m \asymp_\epsilon \log n_m,$$

which is minimal. (This means that there exist $c > 0$ and $C > 0$, possibly depending on ϵ, such that for all m,

$$c \log n_m \leq t_m \leq C \log n_m$$

holds true.)

By this order equality, we say that "*expanders have the best mixing property.*"

In what follows in this section, exercises are on preliminaries of the contents of further sections. They are on elementary groups (over rings) and Banach spaces.

Problem* 10 (Commutator relation on elementary groups)**.** Let $n \geq 2$ and R be an associative ring with unit. For $i, j \in [n](= \{1, 2, \ldots, n\})$ with $i \neq j$ and for $r \in R$, denote by $e_{i,j}^r$ the matrix in $\mathrm{Mat}_{n\times n}(R)$ whose diagonal entries are 1, (i,j)-th entry is r, and all other ones are 0. Recall that the elementary group $\mathrm{E}(n, R)$ is defined as the subgroup of $\mathrm{GL}(n, R)$ generated by $\{e_{i,j}^r : i \neq j \in [n],\ r \in R\}$.

Prove that for *distinct* $i, j, k \in [n]$ and for $r, s \in R$,

$$(\#) \qquad [e_{i,j}^r, e_{j,k}^s] = e_{i,k}^{rs}.$$

Here our convention of group commutators is: $[g, h] = ghg^{-1}h^{-1}$.

Problem 11. Show that ℓ_2 is uniformly convex, and determine the best possible $\delta\colon (0, 2] \to \mathbb{R}_{>0}$ (modulus of convexity).

Problem 12. Let $X = \ell_2\text{-}\bigoplus_{n\geq 2} \ell_{2n}$. This means that

$$X = \left\{ (\xi_n)_{n\geq 2} : \xi_n \in \ell_{2n},\ \sum_{n\geq 2} \|\xi_n\|_{2n}^2 < \infty \right\}$$

with the norm $\|(\xi_n)_n\|_X = \sqrt{\sum_{n\geq 2} \|\xi_n\|_{2n}^2}$.

(1) Is X strictly convex? Is X reflexive?

 (Need not prove in details.)

(2) Show that for every $\epsilon > 0$, all graph families $(\Gamma_m = (V_m, E_m))_m$ (including expander families) admit $(1+\epsilon)$-*biLipschitz embedding* into X, i.e., for each m, there exists $f_m \colon V_m \to X$ such that for all $v, w \in V_m$,

$$\frac{1}{1+\epsilon} d_{\Gamma_m}(v,w) \leq \|f_m(v) - f_m(w)\|_X \leq (1+\epsilon) d_{\Gamma_m}(v,w).$$

 Here d_{Γ_m} denotes the graph metric (shortest path metric) with respect to Γ_m on V_m.

 (Hint: Problem 8.)

Problem 13. Construct a superreflexive Banach space that is not uniformly convex.

(Enflo's characterization of superreflexivility (being isomorphic to a uniformly convex Banach space) can be freely used.)

2. Property (T)

Problem* 14 (Orthogonal complement of invariant part of normal subgroups)**.** Let G be a group, and $N \trianglelefteq G$ be a (closed) *normal* subgroup of G. Let $\pi \colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G.

Show that the orthogonal decomposition associated with $\pi(N)$-invariant vectors, $\mathcal{H} = \mathcal{H}^{\pi(N)} \oplus (\mathcal{H}^{\pi(N)})^{\perp}$ is in fact a decomposition as G-representations. Namely, for all $\xi \in \mathcal{H}^{\pi(N)}$ and for all $\eta \in (\mathcal{H}^{\pi(N)})^{\perp}$, for all g *in* G,

$$\pi(g)\xi \in \mathcal{H}^{\pi(N)} \quad \text{and} \quad \pi(g)\eta \in (\mathcal{H}^{\pi(N)})^{\perp}.$$

Problem 15 (Left regular representation)**.** Let G be a locally compact group and ν be a (left-invariant) Haar measure. Prove that the *left regular representation* $\lambda_G \colon G \to \mathcal{U}(L_2(G,\nu))$,

$$(\lambda_G(g)\xi)(x) = \xi(g^{-1}x), \quad g \in G,\ x \in G,\ \xi \in L_2(G,\nu)$$

is indeed a unitary representation. Here the coefficient field of $L_2(G,\nu)$ is $\mathbb{C}$.

(Show that it is unitary and that it is a group representation.)

Problem* 16. Let $G = \mathbb{R}$, and endow it with the Lebesgue measure. Show that the left regular representation $\lambda = \lambda_{\mathbb{R},1}$ of $\mathbb{R}$ on $L_1(\mathbb{R})$,

$$(\lambda_{\mathbb{R},1}(g)\xi)(x) = \xi(x-g), \quad g \in \mathbb{R},\ x \in \mathbb{R},\ \xi \in L_1(\mathbb{R})$$

is strongly continuous. Namely, whenever $g_m \to g$ in $\mathbb{R}$ as $m \to \infty$, for every $\xi \in L_1(\mathbb{R})$,

$$\|\lambda(g_m)\xi - \lambda(g)\xi\|_1 \to 0 \quad \text{as } m \to \infty.$$

Here the coefficient field of $L_1(\mathbb{R})$ is $\mathbb{R}$.

Problem 17. Is the representation $\lambda = \lambda_{\mathbb{R},1}$ as in Proposition 16 norm continuous? This means, if $g_m \to g$ on $\mathbb{R}$ as $m \to \infty$, does it follow that

$$\|\lambda(g_m) - \lambda(g)\|_{\mathbb{B}(L_1(\mathbb{R}))} \to 0$$

as $m \to \infty$?

Here $\|\cdot\|_{\mathbb{B}(L_1(\mathbb{R}))}$ denotes the *operator norm* on $\mathbb{B}(L_1(\mathbb{R}))$ (the algebra of bounded linear operators on $L_1(\mathbb{R})$), namely, for $T \in \mathbb{B}(L_1(\mathbb{R}))$,

$$\|T\|_{\mathbb{B}(L_1(\mathbb{R}))} = \inf_{\xi \in L_1(\mathbb{R}) \setminus \{0\}} \frac{\|T\xi\|_1}{\|\xi\|_1}.$$

Problem* 18 (Contragredient representation)**.** Let $\rho\colon G \to O(X)$ be an isometric linear representation. Here X is a Banach space and $O(X)$ denotes the group of surjective linear isometries on X. Show that $\rho^\dagger\colon G \to O(X^*)$, defined by for $\phi \in X^*$ and for $\xi \in X$,

$$\langle \rho^\dagger(g)\phi|\xi\rangle = \langle \phi|\rho(g^{-1})\xi\rangle$$

is an isometric linear representation on X^*.

Here X^* means the (continuous) deal of X, and $\langle\cdot|\cdot\rangle\colon X^* \times X \to \mathbb{K}$ is the duality coupling. (Here $\mathbb{K}$ is the coefficient field, $\mathbb{R}$ or $\mathbb{C}$.)

This $\rho^\dagger$ is called the *contragredient representation* of ρ.

Problem* 19 (Contragredient representation and strong continuity)**.** For $\lambda = \lambda_{\mathbb{R},1}$ as in Proposition 16, describe $\lambda^\dagger$. Is $\lambda^\dagger$ strongly continuous?

Problem* 20. Let G be a topological group and $A, B \subseteq G$ be non-empty compact subsets of G. Show that then $AB \subseteq G$ is compact. Here AB denotes the set of all elements of the form $ab(\in G)$, where $a \in A$ and $b \in B$.

Problem 21. Construct a *counterexample* to the following assertion: "*Let G be a compactly generated group and S be a compact generating set of G such that $e_G \in S = S^{-1}$. Then, for every compact subset K of G, there exists $n \in \mathbb{N}$ such that $S^n \supseteq K$.*"

Here S^n denotes the set of all elements that can be written as the product of n elements (possibly overlapping) of S.

(Remark. As explained in the lecture, the assertion above holds true, provided that G is locally compact or Polish.)

Problem* 22 (Property (T) and continuous image of homomorphisms)**.** Let G and H be topological groups, and let $\phi\colon G \to H$ be a continuous homomorphism. Assume that G has property (T).

Verify that then $\overline{\phi(G)}(\subseteq H)$ (the closure of $\phi(G)$ in H) has property (T) (as a topological group in the relative topology from that of H).

Problem* 23 (Property (T) and amenability)**.** Let G be a topological group with property (T), and let A be a locally compact amenable group. Show that then for every continuous homomorphism $\phi\colon G \to A$, the image $\phi(G)$ is relatively compact in A, i.e., $\overline{\phi(G)}$ is compact.

(Remark. The assertion above is *false* if A is not locally compact. One example given by Bekka is the unitary group $\mathcal{U}(\mathcal{H})$ for an infinite dimensional separable Hilbert space $\mathcal{H}$, endowed with the weak operator topology; it has both property (T) and the amenability.)

Problem* 24 (Compact groups have "property (T_X)")**.** Let G be a *compact* group. Let X be a Banach space and $\rho\colon G \to O(X)$ be a strongly continuous isometric linear representation. Assume that there exists $\xi \in X$ such that

$$\sup_{g\in G} \|\rho(g)\xi - \xi\| < \|\xi\|.$$

Show that then $X^{\rho(G)} \neq \{0\}$.

(Hint: Consider a way to construct a $\rho(G)$-invariant vector. Note that this is helpful only if the resulting vector is *non-zero.*)

Moreover, by using the fact proved above, show that *all compact groups have property* (T).

Problem# 25 (Property (T) and finite generation)**.** Prove the following: "*If a discrete group G has property* (T)*, then G is finitely generated.*"

(Hint: If this in full generality is difficult, then first consider the case where G is countable.)

(Remark. A similar argument shows that a locally compact group with property (T) must be compactly generated.)

Problem 26. Let G be a topological group and H be a dense subgroup of G. Prove or disprove the following assertions.

(1) "If H has property (T) as a discrete group, then G has property (T) (as a topological group in the original topology)."
(2) "If G has property (T), then H has property (T) as a discrete group."

Problem 27 (Heredity from a lattice to the original group: discrete case)**.** Let G be a discrete group and H be a finite index subgroup of G. Show that if H has property (T), then so does G.

Problem# 28 (Heredity to lattices: discrete case)**.** Let G be a discrete group and H be a finite index subgroup of G. Show that if G has property (T), then so does H.

(Hint: Complete the outline that is given in the lecture.)

3. Kazhdan constant

Problem* 29. Let G be a topological group, and S_1, S_2 are compact generating subsets of G which are symmetric. Assume that there exists $n \in \mathbb{N}$ such that $S_2^n \supseteq S_1$. Show that for such n,

$$\mathcal{K}(G, S_2) \geq \frac{1}{n}\mathcal{K}(G, S_1).$$

Problem 30 (Kazhdan constant for the pair of compact groups and themselves)**.** Let G be a non-trivial compact group. Show that

$$\mathcal{K}(G, G) \geq \sqrt{2}.$$

(Hint: Modify the argument in Problem 24 for the case where X is a Hilbert space.)

Problem* 31 (Property ($\mathrm{T}_{\mathcal{X}}$))**.** Let G be a topological group and $\mathcal{X}$ be a (non-empty) class of Banach spaces. The notion of *property* ($\mathrm{T}_{\mathcal{X}}$) is defined as follows. (This is a version by M. de la Salle. The original formulation was given by Bader–Furman–Gelander–Monod.)

Definition (Property ($\mathrm{T}_{\mathcal{X}}$))**.** The group G is said to have property ($\mathrm{T}_{\mathcal{X}}$) if the following holds true: For every $X \in \mathcal{X}$ and for every strongly continuous isometric linear representation $\rho\colon G \to O(X)$, there exist a compact subset $K \subseteq G$ and $\epsilon > 0$ such that for all $\xi \in X$,

$$\mathrm{disp}_\rho^K(\xi) \geq \epsilon \|\bar{\xi}\|_{X/X^{\rho(G)}}.$$

Here $\xi \mapsto \bar{\xi}$ is the natural quotient map $X \twoheadrightarrow X/X^{\rho(G)}$, and $\|\cdot\|_{X/X^{\rho(G)}}$ denotes the quotient norm

$$\|\bar{\xi}\|_{X/X^{\rho(G)}} = \inf_{\eta \in X^{\rho(G)}} \|\xi + \eta\|_X.$$

(1) Show that if $\mathcal{X} = \mathcal{H}$ilbert, that is, the class of all Hilbert spaces, then property ($\mathrm{T}_{\mathcal{H}\mathrm{ilbert}}$) coincides with property (T).
(2) Show that $\mathbb{Z}$ has property ($\mathrm{T}_{\mathbb{R}^2}$). Here $\mathbb{R}^2$ means the class consisting only of $\mathbb{R}^2$, which is the 2-dimensional real Hilbert space.

    (Remark. In fact, every group G has property ($\mathrm{T}_{\mathbb{R}^2}$).)
(3) Assume that there exists $p \in [1, \infty)$ such that $\mathcal{X}$ is *closed under taking at most countable ℓ_p-sum.* Namely, for $(X_n)_{n\in\mathbb{N}}$ with $X_n \in \mathcal{X}$, the ℓ_p-direct sum ℓ_p-$\bigoplus_n X_n$ and finite ℓ_p sums ℓ_p-$\bigoplus_{n\leq k} X_n$ for $k \in \mathbb{N}$ all belong to $\mathcal{X}$. Show then the following assertion: "*Assume that G is finitely generated. If G has property* ($\mathrm{T}_{\mathcal{X}}$)*, then for every finite symmetric generating set S of G, the following quantity defined by*

$$\mathcal{K}_{\mathcal{X}}(G, S) = \inf_{\rho} \inf_{\xi \in X \setminus X^{\rho(G)}} \frac{\mathrm{disp}_\rho^S(\xi)}{\|\bar{\xi}\|_{X/X^{\rho(G)}}}$$

    is strictly positive." Here ρ runs over all strongly continuous isometric linear G-representation on X, where $X \in \mathcal{X}$, such that $X^{\rho(G)} \neq X$. (If there is no such ρ, then define $\mathcal{K}_{\mathcal{X}}(G, S)$ as $+\infty$.)

(Remark. If G is discrete, then it is automatic that G above is finitely generated under the assumption above on $\mathcal{X}$. Prove this if you are interested in it.)

This quantity $\mathcal{K}_{\mathcal{X}}(G, S)$ may be said as the *Kazhdan constant for* (G, S) *with respect to* $\mathcal{X}$.

(4) Construct a *counterexample* to the assertion of (3) when we drop the condition on $\mathcal{X}$ on the closedness by taking ℓ_p-sums.

Problem* 32 (Property $(\mathrm{T}_{\mathcal{X}})$ and Banach (X, p)-anders)**.** Let $\mathcal{X}$ be a (non-empty) class of Banach spaces and $p \in [1, \infty)$. Assume that $\mathcal{X}$ is closed under taking at most countable ℓ_p-sum. Show that then the *following analog of Margulis's theorem* holds true:

"Let G *be a finitely generated group and* S *be a finite symmetric generating set of* G*. Assume that* G *has property* $(\mathrm{T}_{\mathcal{X}})$ *and that there exists a family of surjective homomorphisms* $(\phi_m \colon G \twoheadrightarrow G_m)_{m \in \mathbb{N}}$ *such that* $|G_m| < \infty$ *and* $\lim_{m\to\infty} |G_m| = \infty$*. Then for every* $X \in \mathcal{X}$*, the family* $(\Gamma_m = \mathrm{Cay}(G_m, \phi_m(S)))_{m\in\mathbb{N}}$ *forms that of Banach* (X, p)*-anders."*

The following exercises in this section are on relative property (T).

Problem 33. As also mentioned in the lecture, it is well-known that the pair

$$\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$$

has relative property (T). Here $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathbb{Z}^2$ by natural matrix multiplications. Show that $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ itself does not have property (T).

Problem# 34 (Property (T) and group extensions)**.** Let

$$1 \to N \overset{\iota}{\to} G \overset{q}{\to} Q \to 1$$

be a short exact sequence among locally compact groups. Through ι, we regard N as a closed normal subgroup of G. Prove the following assertions.

(1) If G has property (T), then $G \trianglerighteq N$ has relative property (T) and Q has property (T).
(2) Conversely, if $G \trianglerighteq N$ has relative property (T) and if Q has property (T), then G has property (T).

4. Property $(\mathrm{F}_{\mathcal{X}})$

Problem* 35 (Affine isometric actions and cocycles)**.** Let G be a group, and X be a Banach space. Let $\alpha \colon G \curvearrowright X$ be an affine action of G on X, that means, for each $g \in G$, $\alpha(g)$ is of the form

$$\alpha(g) \cdot \xi = \rho(g)\xi + b(g) \quad \text{for } \xi \in X,$$

where $\rho(g) \in \mathbb{B}(X)$ and $b(g) \in X$; furthermore, for all $g, h \in G$,

$$\alpha(g) \cdot \alpha(h) = \alpha(gh).$$

Assume besides that α is isometric, namely, for every $g \in G$ and for all $\xi, \eta \in X$,

$$\|\alpha(g) \cdot \xi - \alpha(g) \cdot \eta\| = \|\xi - \eta\|.$$

(1) Prove that for all $g \in G$, $\rho(g)$ is isometric.
(2) Prove that $\rho\colon G \to O(X)$ is a representation (i.e., it is a group homomorphism).

(Hint: Suppose that it is not the case. Then there exist $g, h \in G$ and $\xi \in X$ such that $\rho(g)\rho(h)\xi - \rho(gh)\xi \neq 0$. Draw a contradiction from this.)
(3) Prove that $\alpha\colon G \to X$ is a *ρ-cocycle*. Namely, for all $g, h \in G$,

$$b(gh) = b(g) + \rho(g)b(h).$$

Problem* 36. Let $\rho\colon G \to O(X)$ be an isometric linear representation of a group G on a Banach space X. Let $\rho\colon G \to X$ be a ρ-cocycle. Prove the following formulae.
(1) $b(e_G) = 0$.
(2) For all $g \in G$, $b(g^{-1}) = -\rho(g)^{-1}b(g)$.
(3) For all $n \in \mathbb{N}_{\geq 2}$ and for all $g_1, \ldots, g_n \in G$,

$$b(g_1 \cdots g_n) = b(g_1) + \sum_{k=2}^{n} \rho(g_1 \cdots g_{k-1})b(g_k).$$

In particular,

$$\|b(g_1 \cdots g_n)\| \leq \sum_{k=1}^{n} \|b(g_k)\|.$$

Problem 37. Let G be a topological group, and $G' \geqslant H' \geqslant H$ be closed subgroups of G. Let $\mathcal{X}$ be a (non-empty) class of Banach spaces. Show that $G \geqslant H$ has relative property ($\mathrm{F}_{\mathcal{X}}$) if $G' \geqslant H'$ has relative property ($\mathrm{F}_{\mathcal{X}}$).

Problem 38 (Property ($\mathrm{F}_{\mathcal{X}}$) and finite generation)**.** Let $\mathcal{X}$ is a (non-empty) class of Banach spaces that is closed under taking at most countable ℓ_p-sums for some $p \in [1, \infty)$. Assume that G is a *countable* discrete group. Show that if G has property ($\mathrm{F}_{\mathcal{X}}$), then G is finitely generated.

In particular, if such G has property ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$), then G is finitely generated.

(Remark. The conclusion is known to be *false* if we drop the countability assumption on G. Compare with Problem 25. For instance, it is known that the group $\mathrm{Sym}_{\mathrm{full}}(\mathbb{N})$ of *all* bijections on $\mathbb{N}$ has property ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$) as a discrete group. For the proof, see Bergman, "*Generating infinite symmetric groups*" and de Cornulier, "*Strongly Bounded Groups and Infinite Powers of Finite Groups*".)

Problem 39. Construct a *counterexample* to the assertion of Problem 38 if we drop the closedness assumption on $\mathcal{X}$ by taking ℓ_p-sums.

Problem* 40 (Another proof of the Guichardet's theorem: discrete group case)**.** Let G be a countable discrete group. Here we will prove the Guichardet's direction

"*G fails to have* (T) $\implies$ *G fails to have* ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$)"

in a different manner to one exhibited in the lecture.

(1) Show that we may assume that G is finitely generated.

Therefore, we assume so, and fix a finite symmetric generating set S of G. Suppose that G fails to have (T). Then for every $n \in \mathbb{N}$, there exist a unitary representation $\pi_n \colon G \to \mathcal{U}(\mathcal{H}_n)$ and a vector $\xi_n \in \mathcal{H}_n \setminus \mathcal{H}_n^{\pi_n(G)}$ such that

$$\mathrm{disp}^S_{\pi_n}(\xi_n) < 2^{-2n}\|\overline{\xi_n}\|_{\mathcal{H}_n/\mathcal{H}_n^{\pi_n(G)}}.$$

(2) Show that for every n, we may assume that $\|\overline{\xi_n}\|_{\mathcal{H}_n/\mathcal{H}_n^{\pi_n(G)}} = 2^n$.

(3) Construct a Hilbert space $\mathcal{H}$ and an affine isometric action $\alpha \colon G \curvearrowright \mathcal{H}$ of G on $\mathcal{H}$ that has no global fixed point.

This shows that G fails to have ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$), as desired.

(Remark. This proof extends to that of

$$\text{Property } (\mathrm{F}_{\mathcal{X}}) \quad \Longrightarrow \quad \text{property } (\mathrm{T}_{\mathcal{X}})$$

for countable discrete groups, provided that $\mathcal{X}$ is stable under taking at most countable ℓ_p-sum for some $p \in [1, \infty)$. This assumption on $\mathcal{X}$ may be removed if we argue in the way described in the lecture. However, the current proof may be generalized to more general situation, for instance, fixed point properties with respect to certain classes of non-linear metric spaces.)

Problem* 41 (Lemma of the Chebyshev center)**.** Let X be a *dual* Banach space (i.e., isometrically isomorphic to the dual Y^* of some Banach space Y). Let $A \subseteq X$ be a non-empty *bounded* subset of X.

(i) For each $\xi \in X$, define

$$R_\xi = \sup_{a \in A} \|\xi - a\|$$

and define $R = \inf_{\xi \in X} R_\xi$.

We will show that R is *attained*, namely, there exists $\xi \in X$ such that $R_\xi = R$.

(1) Fix $\xi \in X$. Show the following equivalence for $T > 0$:

$$T > R_\xi \quad \Longleftrightarrow \quad \xi \in \bigcap_{a \in A} B(a, T).$$

Here $B(\eta, r)$ denotes the closed ball of radius r centered at η in X.

(2) Show that

$$C_A = \{\xi \in X : R_\xi = R\}$$

is non-empty.

(Hint: Use compactness in a nice way. Which topology can we use?)

(ii) Assume besides that X is uniformly convex. Show that then C_A is a singleton, i.e., C_A consists of exactly one point.

Note that if X is uniformly convex, then X is reflexive and in particular a dual Banach space. In that case, the *unique* point in C_A is called the *Chebyshev center* of A in X.

Problem 42. Construct a *counterexample* to the conclusion of (*ii*) in Problem 41 when we drop the uniform convexity assumption on the dual Banach space X.

Problem* 43. Let G be an (arbitrary) group and X be a *uniformly convex* Banach space. Let $\rho\colon G \to O(X)$ is an isometric linear representation. Assume that there exists $\xi \in X$ such that

$$\sup_{g\in G} \|\rho(g)\xi - \xi\| < \|\xi\|.$$

Show that then $X^{\rho(G)} \neq \{0\}$.

(Remark. Compare with Problem 24.)

Problem 44. Let $\ell_{1,0} := \{\xi \in \ell_1(\mathbb{N}) : \sum_n \xi(n) = 0\}$. Then as we see in the lecture, every countable infinite (discrete) group G admits an affine isometric action on $\ell_{1,0}$ without global fixed points such that orbits are bounded.

Explain what is wrong if we consider $\ell_{2,0} := \{\xi \in \ell_2(\mathbb{N}) : \sum_n \xi(n) = 0\}$ and if we would deduce that same construction can be made for actions on Hilbert spaces. (By Problem 41, this deduction must be wrong.)

Problem 45 (Niblo–Reeves theorem)**.** Extend the argument of Watatani's theorem on groups acting on simplicial trees, and prove the following theorem.

Theorem . *Let G be a discrete group. Let Y be a* CAT(0) *cube complex. Let $\beta\colon G \curvearrowright Y$ be a cellular action, namely, an action by cellular automorphisms. Fix a vertex v_0 in $Y^{(0)}$.*

Then, for every $p \in [1,\infty)$, there exists an affine isometric action

$$\alpha_p\colon G \curvearrowright \ell_p$$

on ℓ_p that satisfies the following condition on the orbit of the origin $0 \in \ell_p$: For every $g \in G$,

$$\|\alpha_p(g)\cdot 0\|_p = (2d_1(v_0, \beta(g)\cdot v_0))^{1/p}.$$

Here, d_1 means the ℓ_1-metric on $Y^{(0)}$, namely, the shortest path metric based on the graph being the 1*-skelton $Y^{(1)}$ of Y.*

From this, draw the following conclusion: "*If a discrete group G acts on a finite dimensional* CAT(0) *cube complex Y cellularly without fixed points* (*on Y*)*, then for all $p \in [1,\infty)$, G fails to have property* (F_{ℓ_p})."

(Remark. Later, de Cornulier extended the last conclusion above to the case where Y is infinite dimensional. See arXiv:1302.5982 for details.)

Problem$^\#$ 46 (Gurchardet's theorem for relative properties)**.** Extend the proof of Guichardet's direction ("($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$)$\Rightarrow$(T)") to relative case and prove the following.

"*Let G be a finitely generated group and H is a normal subgroup of G. Then, $G \trianglerighteq H$ has relaive property* (T) *if it has relative property* ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$)."

(Remark. As mentioned in the lecture, the proof here may be further extended to the case where G is countable (discrete) by means of the open mapping theorem for Fréchet spaces.)

Problem$^{\#}$ 47 (Delorme's theorem for discrete groups)**.** We will show the Delorme's direction

$$\text{“}\textit{Property}\ (\mathrm{T}) \quad \Longrightarrow \quad \textit{Property}\ (\mathrm{F}_{\mathcal{H}\mathrm{ilbert}})\text{”}$$

for discrete groups.

For this, we briefly give the definition of tensor products of (real) Hilbert spaces. Let $\mathcal{H}_1, \mathcal{H}_2$ be real Hilbert spaces. Then, equip the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}_1} \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_2}$$

($\xi_1, \xi_2 \in \mathcal{H}_1$ and $\eta_1, \eta_2 \in \mathcal{H}_2$). This means, we extend the definition above to real linear combinations by linearity (this definition of the inner product is well-defined). Finally, take the completion of the algebraic tensor product in the norm given by

$$\|\Xi\| = \sqrt{\langle \Xi, \Xi \rangle}$$

(for Ξ in the algebraic tensor product). This completed space is equipped with the inner product structure defined as above, and hence is a real Hilbert space. We simply write this resulting space as $\mathcal{H}_1 \otimes \mathcal{H}_2$, and call it the (real) *Hilbert space tensor product*.

Now let $\mathcal{H}$ be a real Hilbert space. Then, define

$$\mathrm{Exp}(\mathcal{H}) = \mathbb{R} \oplus_{\ell_2} \mathcal{H} \oplus_{\ell_2} \mathcal{H} \otimes \mathcal{H} \oplus_{\ell_2} \mathcal{H}^{\otimes 3} \oplus_{\ell_2} \mathcal{H}^{\otimes 4} \oplus_{\ell_2} \cdots,$$

where $\oplus_{\ell_2}$ means the ℓ_2-sum, and $\mathcal{H}^{\otimes n}$ denotes the n-time tensor product (this is well-defined). (This space $\mathrm{Exp}(\mathcal{H})$ is a real Hilbert space, and sometimes called the (real) *full Fock space*.) For each $\xi \in \mathcal{H}$, set

$$\mathrm{Exp}(\xi) = 1 \oplus \xi \oplus \frac{1}{\sqrt{2!}} \xi \otimes \xi \oplus \frac{1}{\sqrt{3!}} \xi \otimes \xi \otimes \xi \oplus \frac{1}{\sqrt{4!}} \xi \otimes \xi \otimes \xi \otimes \xi \oplus \cdots$$

and

$$\exp(\xi) = \exp\left(-\frac{\|\xi\|^2}{2}\right) \mathrm{Exp}(\xi) \in \mathrm{Exp}(\mathcal{H}).$$

(1) For all $\xi, \eta \in \mathcal{H}$, show that $\|\exp(\xi)\| = 1$ and that

$$\langle \exp(\xi), \exp(\eta) \rangle = \exp\left(-\frac{\|\xi - \eta\|^2}{2}\right).$$

Set $\widehat{\mathcal{H}}_0$ is the algebraic (real) span of $\{\exp(\xi) : \xi \in \mathcal{H}\}$, which is a (real) possibly non-closed subspace of $\mathrm{Exp}(\mathcal{H})$. Define $\widehat{\mathcal{H}}$ as the closure of $\widehat{\mathcal{H}}_0$ inside $\mathrm{Exp}(\mathcal{H})$. This is a closed subspace of $\mathrm{Exp}(\mathcal{H})$, and hence is a real Hilbert space.

(2) Let $\alpha \colon G \curvearrowright \mathcal{H}$ is an affine isometric action. For each $g \in G$, define a (linear) operator $\pi_{\alpha,0}(g)$ on $\widehat{\mathcal{H}}_0$ by

$$\pi_{\alpha,0}(g)\left(\sum_k c_k \exp(\xi_k)\right) = \sum_k c_k \exp(\alpha(g) \cdot \xi_k).$$

Here the sum above is a finite sum and $c_k \in \mathbb{R}$.

Show the following.

(a) For all $g, h \in G$, $\pi_{\alpha,0}(g)\pi_{\alpha,0}(h) = \pi_{\alpha,0}(gh)$.

(b) For all $g \in G$, $\pi_{\alpha,0}(g)$ is an isometric (linear) operator on $\widehat{\mathcal{H}}_0$.
(Strictly speaking, the well-defindness of $\pi_{\alpha,0}$ is ensured only after we prove (b).)

By (b), for every $g \in G$, $\pi_{\alpha,0}(g)$ uniquely extends to an operator

$$\pi_\alpha(g) \colon \widehat{\mathcal{H}} \to \widehat{\mathcal{H}},$$

which is orthogonal (real unitary). By (a), this gives an orthogonal representation

$$\pi_\alpha \colon G \to O(\widehat{\mathcal{H}}).$$

(3) Assume besides that $\mathcal{H}^{\alpha(G)} = \emptyset$. Show that then

$$\pi_\alpha \not\supseteq 1_G.$$

(4) Complete the proof of Delorme's direction. More precisely, suppose that $\mathcal{H}^{\alpha(G)} = \emptyset$ in the setting as in (2). Then, construct a unitary representation ρ on some Hilbert space $\mathfrak{H}$ such that $\rho \not\supseteq 1_G$ but $\rho \succeq 1_G$.

(Hint. In general, it is not clear whether the π_α above weakly contains 1_G. Instead, we can create a new representation out of π_α's. If you are careful, then you will notice that resulting representation is (real) orthogonal but not (complex) unitary. If you hope to modify this small issue, then there is a procedure called "complexification" of orthogonal representations to unitary ones.)

Problem$^{\#}$ 48 (Haagerup property and a-T-menability)**.** Let G be a countable discrete group. The following properties can be seen as extreme negations (among non-compact groups), respectively, to property (T) and to property ($\mathrm{F}_{\mathcal{H}\mathrm{ilbert}}$).

Definition . (1) The group G is said to have the *Haagerup property* if there exists a unitary C_0-representation $\pi \colon G \to \mathcal{U}(\mathcal{H})$ such that $\pi \succeq 1_G$. Here a unitary representation $\rho \colon G \to \mathcal{U}(\mathfrak{H})$ is said to be C_0 if for all $\xi, \eta \in \mathfrak{H}$,

$$\langle \rho(g)\xi, \eta \rangle \to 0 \quad \text{as } g \to \infty.$$

Here "$g \to \infty$" means g moves in any direction with escaping from every compact (finite) subset of G. More precisely, "$\lim_{g\to\infty}\langle \rho(g)\xi, \eta\rangle = 0$" means that for every $\epsilon > 0$, the set

$$\{g \in G : |\langle \rho(g)\xi, \eta\rangle| \geq \epsilon\}$$

is relatively compact (finite).

(2) The group G is said to be *a*-T-*menable* if there exists a *metrically proper* affine isometric action of G on some Hilbert space. Here, an isometric action $\alpha \colon G \curvearrowright \mathfrak{H}$ is said to be metrically proper if for some (equivalently, for every) $\xi \in \mathfrak{H}$,

$$\|\alpha(g) \cdot \xi\| \to \infty \quad \text{as } g \to \infty.$$

More precisely, this means that for every $C > 0$, the set

$$\{g \in G : \|\alpha(g) \cdot \xi\| \leq C\}$$

is relatively compact (finite).

Show that (for countable discrete groups), the Haagerup property and the a-T-manebility are equivalent.

Problem* **49.** Show that $\mathbb{Z}$ has the Haagerup property by showing it with the original definition and by showing the a-T-manebility.

Problem* **50.** Show that F_2 (the non-abelian free group of rank 2) is a-T-maneble.

Problem* **51.** Show that $\mathrm{SL}(2,\mathbb{Z})$ is a-T-maneble.

Problem* **52** (Properties of Haagerup property/a-T-maneblity 1)**.** (1) Let G be a countable (discrete) group that is a-T-maneble. For a (closed) subgroup H of G, does it follow that H is also a-T-maneble?

(2) Let G be a countable (discrete) group that is a-T-maneble. For a group quotient Q of G, does it follow that Q is also a-T-maneble?

Problem* **53** (Properties of Haagerup property/a-T-maneblity 2)**.** Prove or disprove the following assertion: "*The a-*T*-manebility for countable* (*discrete*) *groups is closed under group extension. Namely, for a short exact sequence among countable groups,*

$$1 \to N \to G \to Q \to 1,$$

*if N and Q are a-*T*-maneble, then so is G.*"

5. Property ($\mathrm{T}_{\mathcal{X}}$) and property ($\mathrm{F}_{\mathcal{X}}$)

Problem* **54** (Isometric renorming with respect to uniformly bounded representations)**.** Let X be a Banach space. Let G be a group, and $\rho\colon G \to \mathrm{GL}(X)$ be a *uniformly bounded* linear representation, that means, there exists $C > 0$ such that for all $g \in G$,

$$\|\rho(g)\|_{\mathbb{B}(X)} \leq C.$$

Prove that the following norm $\|\cdot\|_\rho$

$$\|\xi\|_\rho = \sup_{g\in G} \|\rho(g)\xi\|_X, \quad \xi \in X,$$

indeed gives a new norm on X, and that $\|\cdot\|_\rho$ is equivalent to $\|\cdot\|_X$, that means, the (set theoretical) identity map

$$\mathrm{id}_X\colon (X, \|\cdot\|_X) \to (X, \|\cdot\|_\rho)$$

gives an isomorphism (not necessarily isometric) between Banach spaces. (This map is bijective linear map, and so the only issue is on continuity.) Moreover, show that ρ is *isometric* in the norm $\|\cdot\|_\rho$.

Therefore, in the setting above, there exist a Banach space Y that is isomorphic to X and a bijective linear operator $T\colon X \to Y$ such that T and T^{-1} are continuous and the representation given by

$$\sigma = T \circ \rho \circ T^{-1}$$

is *isometric.*

Problem* 55. Prove the following: "*If* $(X, \|\cdot\|_X)$ *is uniformly convex in the setting of Problem* 54*, then the resulting norm* $\|\cdot\|_\rho$ *is again uniformly convex.*"

Problem* 56 (Renorming of superreflexive Banach spaces)**.** Let $X = (X, \|\cdot\|_X)$ be a superreflexive Banach space. Let G be a group, and $\rho\colon G \to \mathrm{GL}(X)$ is a uniformly bounded linear representation of G on X.

We will prove that X admits an equivalent renorming $\|\cdot\|_{\mathrm{ucus}}$ such that

- ρ is isometric in $\|\cdot\|_{\mathrm{ucus}}$.
- $(X, \|\cdot\|_{\mathrm{ucus}})$ is uniformly convex and *uniformly smooth*.

Here a Banach space Y is uniformly smooth if and only if its dual Y^* is uniformly convex.

Let $\mathcal{N}(X)$ be the set of all norms on X that are equivalent to $\|\cdot\|_X$. Equip $\mathcal{N}(X)$ with the metric defined by

$$d(\|\cdot\|, \|\cdot\|') = \sup_{\xi \in X\setminus\{0\}} \log\left|\frac{\|\xi\|'}{\|\xi\|}\right|.$$

(It is not difficult to see that this d is a genuine metric on $\mathcal{N}(X)$, and that $(\mathcal{N}(X), d)$ is complete.)

(1) Set a subset $\mathcal{N}_{\mathrm{uc}}(X)^\rho \subseteq \mathcal{N}(X)$ as the set of all norms $\|\cdot\|$ in $\mathcal{N}_X$ such that $(X, \|\cdot\|)$ is uniformly convex and that ρ is isometric in $\|\cdot\|$.

Show that $\mathcal{N}_{\mathrm{uc}}(X)^\rho$ is non-empty.

(Hint. Recall Enflo's characterization of the supereflexivity.)

(2) Show that $\mathcal{N}_{\mathrm{uc}}(X)^\rho$ is in fact a dense subset of $\mathcal{N}(X)^\rho$. Here $\mathcal{N}(X)^\rho$ is the set of all norms in $\mathcal{N}(X)$ with respect to which ρ is isometric.
(Hint. To each norm in $\mathcal{N}(X)^\rho$, deform $\|\cdot\|$ in $\mathcal{N}_{\mathrm{uc}}(X)^\rho$ and make it close to that.)

(3) Set a subset $\mathcal{N}_{\mathrm{us}}(X)^\rho \subseteq \mathcal{N}(X)$ as the set of all norms $\|\cdot\|$ in $\mathcal{N}_X$ such that $(X, \|\cdot\|)$ is uniformly smooth and that ρ is isometric in $\|\cdot\|$.

Show that $\mathcal{N}_{\mathrm{uc}}(X)^\rho \cap \mathcal{N}_{\mathrm{us}}(X)^\rho \neq \emptyset$.

Therefore, in the setting above, there exist a *uniformly convex and uniformly smooth* Banach space Y that is isomorphic to X and a bijective linear operator $T\colon X \to Y$ such that T and T^{-1} are continuous and the representation given by

$$\sigma = T \circ \rho \circ T^{-1}$$

is *isometric.*

Problem 57 (Duality mapping between unit spheres)**.** Let X be a uniformly smooth Banach space. Then X^* is uniformly convex.

(1) Prove the following: "*For every unit vector* $\xi \in X$*, there exists a* unique *norm* 1 *element* ϕ *in* X^* *such that*

$$\langle \phi | \xi \rangle = 1."$$

Here $\langle\cdot|\cdot\rangle\colon X^* \times X \to \mathbb{K}$ is the duality coupling ($\mathbb{K}$ is the coefficient field).

We write the unique element ϕ as ξ^*, and the map

$$S(X) \to S(X^*) \colon \xi \mapsto \xi^*$$

is sometimes called the *duality mapping*. Here $S(Y)$ means the unit sphere of a Banach space Y.

(2) Let $\rho \colon G \to O(X)$ be an isometric linear representation on X of a group G. Then show that for all $\xi \in S(X)$ and for all $g \in G$,

$$(\rho(g)\xi)^* = \rho^\dagger(g)\xi^*.$$

Here $\rho^\dagger \colon G \to O(X^*)$ is the contragredient representation of ρ (recall the definition from Problem 18).

Problem* 58 (Natural complement of invariant vectors in superreflexive Banach spaces)**.** Let X be a superreflexive Banach space. Let G be a group and $\rho \colon G \to \mathrm{GL}(X)$ be a uniformly bounded linear representation. By Problem 56, by taking an equivalent norm if necessary, we may assume that X is uniformly convex and uniformly smooth and that $\rho \colon G \to O(X)$ is isometric.

Under the assumption above, we will show that there exists a natural complement $X'_{\rho(G)}$ of $X^{\rho(G)}$:

$$X = X^{\rho(G)} \oplus X'_{\rho(G)}$$

as G-representations. Furthermore, the projection from X onto $X^{\rho(G)}$ has norm 1, i.e., for all $\xi \in X^{\rho(G)}$ and for all $\eta \in X'_{\rho(G)}$,

$$\|\xi\| \leq \|\xi + \eta\|.$$

Define $X'_{\rho(G)}$ to be the annihilator of $(X^*)^{\rho^\dagger(G)}$ in X, that means,

$$X'_{\rho(G)} = \{\eta \in X : \langle \phi | \eta \rangle = 0 \quad \text{for all } \phi \in (X^*)^{\rho^\dagger(G)}\}.$$

(1) Show that $X'_{\rho(G)}$ is $\rho(G)$-invariant, that means, for all $\eta \in X'_{\rho(G)}$ and for all $g \in G$, $\rho(g)\eta \in X'_{\rho(G)}$ holds.

(2) Show that for all $\xi \in S(X^{\rho(G)})$ and for all $\eta \in X'_{\rho(G)}$

$$\|\xi + \eta\| \geq 1.$$

From this, show that $X^{\rho(G)} \cap X'_{\rho(G)} = \{0\}$ and that $X^{\rho(G)} \oplus X'_{\rho(G)}$ is a *closed* subspace of X.

(3) Show that $X^{\rho(G)} \oplus X'_{\rho(G)} = X$.

(Hint. Suppose that it is not the case. Then, apply the Hahn–Banach separation theorem, and draw the contradiction.)

Problem$^\#$ 59 (Kazhdan projection)**.** We here treat finitely generated groups. Let G be a finitely generated group, and S be a finite symmetric generating set. Let $\mathcal{X}$ be a class of (complex) Banach spaces.

Recall that the *complex group algebra* $\mathbb{C}[G]$ is the ring of elements of the form $f = \sum_g f_g g$, where $f_g \in \mathbb{C}$ and $f_g = 0$ all but finite $g \in G$, equipped with the convolution as the multiplication in the ring. Note that for each linear G-representation $\rho\colon G \to \mathbb{B}(X)$ on a Banach space X (not necessarily isometric), ρ induces the following algebra homomorphism $f \mapsto f_\rho$:

$$\mathbb{C}[G] \ni f = \sum_g f_g g \mapsto f_\rho = \sum_g f_g \rho(g) \in \mathbb{B}(X).$$

Definition . Let $\|\cdot\|_{\mathcal{H}\mathrm{ilbert}}$ be the *maximal norm* on $\mathbb{C}[G]$ *for all unitary representations of* G. That means, $\|f\|_{\mathcal{H}\mathrm{ilbert}} := \sup_{(\pi,\mathcal{H})} \|f_\pi\|_{\mathbb{B}(\mathcal{H})}$, where $(\pi,\mathcal{H})$ runs over all unitary representations of G. The Banach algebra $C_{\mathcal{H}\mathrm{ilbert}}(G)$ is defined as the completion of $\mathbb{C}[G]$ in the norm $\|\cdot\|_{\mathcal{H}\mathrm{ilbert}}$.

In this case, the Banach algebra $C_{\mathcal{H}\mathrm{ilbert}}(G)$ is naturally equipped with the structure of a C^*-algebra. In the standard literature, $\|\cdot\|_{\mathcal{H}\mathrm{ilbert}}$ is written as $\|\cdot\|_{\max}$; $C_{\mathcal{H}\mathrm{ilbert}}(G)$ is written as $C^*(G)$ or $C^*_{\max}(G)$, and called the *maximal* (or *full*) *group* C^**-algebra*.

(1) Set

$$\Delta = \frac{1}{2}\sum_{s\in S}(1-s^{-1})(1-s) \left(= |S| - \sum_{s\in S} s\right) \in \mathbb{C}[G].$$

and

$$a \left(= 1 - \frac{1}{2|S|}\Delta\right) = \frac{1}{2} + \frac{1}{2|S|}\sum_{s\in S} s \in \mathbb{C}[G].$$

Now assume that G has property (T). Show that then $(a^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^*(G)(= C_{\mathcal{H}\mathrm{ilbert}}(G))$.

(2) Prove the following. "*If G has property* (T)*, then there exists an element p in $C^*(G)$ that satisfies the following three conditions:*
  - (*i*) *p is an idempotent, i.e. $p^2 = p$.*
  - (*ii*) *p is the norm limit of elements of the form $f_n = \sum_g f_{n,g} g \in \mathbb{C}[G]$ such that $\sum_g f_{n,g} = 1$, $f_{n,g} \geq 0$, and $f_{n,g^{-1}} = f_{n,g}$.*
  - (*iii*) *For every* (*unitary G-representation*) $(\pi,\mathcal{H})$*, p_π* (*defined as the* (*norm*)*-limit of $(f_{n,\pi})_{n\geq 1}$*) *is a projection onto $\mathcal{H}^{\pi(G)}$.*

  Such p is called a *Kazhdan-type projection.*

(3) Prove the converse. More precisely, show that if a finitely generated group G admits a Kazhdan-type projection in $C^*(G)$, then G has property (T).

Remark. For a general $\mathcal{X}$, we can define the Banach algebra $C_{\mathcal{X}}(G)$ as the completion of $\mathbb{C}[G]$ in the maximal norm for all linear *isometric* representation (ρ, X), $X \in \mathcal{X}$ of G. A *Kazhdan-type projection* in $C_{\mathcal{X}}(G)$ is defined in the same way, except that in (*iii*), $(\pi,\mathcal{H})$ is replaced with (ρ, X) above. We say that G has *property* $(\mathrm{T}^{\mathrm{proj}}_{\mathcal{X}})$ if $C_{\mathcal{X}}(G)$ admits a Kazhdan-type projection.

Further strengthening have been defined and studied by several mathematicians, as follows.

- (V.Lafforgue) *Strong property* $(\mathrm{T}_{\mathcal{X}})$ for G is "roughly" defined as the existence of a Kazhdan-type projection in the Banach algebra $C_{\mathcal{X},C,D}(G)$ for some $C > 0$ and all $D \geq 1$. Here $C_{\mathcal{X},C,D}(G)$ is defined as the completion of $\mathbb{C}[G]$ in the maximal norm for all linear representations (ρ, X), $X \in \mathcal{X}$ of G that satisfy
$$\text{for all } g \in G, \quad \|\rho(g)\|_{\mathbb{B}(X)} \leq De^{C|g|_S},$$
where $|\cdot|_S$ is the word length on G with respect to S.
- (Oppenheim) *Robust property* $(\mathrm{T}_{\mathcal{X}})$ for G is defined as the existence of a Kazhdan-type projection in the Banach algebra $C_{\mathcal{X},D}(G)$ for some $D > 1$. Here $C_{\mathcal{X},D}(G)$ is defined as the completion of $\mathbb{C}[G]$ in the maximal norm for all linear representations (ρ, X), $X \in \mathcal{X}$ of G that satisfy $\max_{s\in S} \|\rho(s)\|_{\mathbb{B}(X)} \leq D$.

Problem 60. Let G be a finitely generated group and $\mathcal{X}$ be a (non-empty) class of Banach spaces. Show that if G has property $(\mathrm{T}_{\mathcal{X}}^{\mathrm{proj}})$, then G has property $(\mathrm{T}_{\mathcal{X}})$ *in the uniform way*, that means, for every (equivalently, some) finite symmnetric generating set S of G,
$$\mathcal{K}_{\mathcal{X}}(G, S) > 0.$$
Here $\mathcal{K}_{\mathcal{X}}(G, S)$ is defined in Problem 31.

(In the literature, the last property is called "*uniform property* $(\mathrm{T}_{\mathcal{X}})$.")

Problem 61. Prove that if a countable discrete group G has property $(\mathrm{T}_{\ell_1}^{\mathrm{proj}})$, then G is finite.

Therefore, the converse to Problem 60 is *false* in general.

(Remark. Druţu and Nowak (arXiv:1501.03473) showed that the converse of Problem 60 holds true if $\mathcal{X}$ consists of superreflexive Banach spaces.)

6. Main byproducts of Main Theorems

Problem 62. Let R be a unital, finitely generated (associative) ring, and $n \geq 3$. By following the lecture, set $G = \mathrm{E}(n, R)$ and

- $M = \langle e_{i,n}^r : i \in [n-1], r \in R\rangle (\simeq (R^{n-1}, +))$,
- $L = \langle e_{n,j}^r : j \in [n-1], r \in R\rangle (\simeq (R^{n-1}, +))$.

Show that $\langle M, L\rangle = G$.

Problem 63 (Cartan-type involution)**.** Denote by $R = \mathbb{Z}\langle x_1, \ldots, x_k\rangle$ $(k \in \mathbb{N})$ the non-commutative polynomial ring over $\mathbb{Z}$. Define a map $\tau \colon R \to R$ by setting $\tau(1) = 1$ and $\tau(x_i) = x_i$ for $i \in [k]$, and by extending it in multiplication-reversing way: $\tau(r_1r_2) = \tau(r_2)\tau(r_1)$ for $r_1, r_2 \in R$. In this way, τ gives rise to an *orientation-reversing* ring isomorphism.

Show that for every $n \in \mathbb{N}_{\geq 2}$, the following map
$$\mathrm{E}(n, R) \to \mathrm{E}(n, R); \quad (r_{i,j})_{i,j} \mapsto (\tau(r_{j,i}))_{i,j}$$
is a group isomorphism.

7. Metric ultraproducts

Problem 64. Show that every free ultrafilter (also known as, non-principal ultrafilter) on $\mathbb{N}$ contains the cofinite filter on $\mathbb{N}$.

Problem* 65. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Prove that for every bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, $\lim_{\mathcal{U}} a_n$ exists and that it is unique.

Problem* 66. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For each $k \in \mathbb{N}$, let $(a_n^{(k)})_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Prove the following. Here $\mathcal{U}$-limits are taken for n.

(1) If $\lim_{n\to\infty} a_n^{(1)}$ exists, then $\lim_{\mathcal{U}} a_n^{(1)} = \lim_{n\to\infty} a_n^{(1)}$.
(2) For each k and for every $t_1, \ldots, t_k \in \mathbb{R}$,

$$\lim_{\mathcal{U}} \left(\sum_{i=1}^{k} t_i a_n^{(i)} \right) = \sum_{i=1}^{k} (t_i \lim_{\mathcal{U}} a_n^{(i)}).$$

(3) For each k,

$$\lim_{\mathcal{U}} \sup_{1\leq i\leq k} a_n^{(i)} = \sup_{1\leq i\leq k} \lim_{\mathcal{U}} a_n^{(i)}.$$

Similarly on inf.

(Hint: For sup, the real problem is to show "$\leq$".)

Problem 67. Construct a *counterexample* to (3) in Problem 66 when $k = \infty$. (Construct one that satisfies $\sup_{n,k} |a_n^{(k)}| < \infty$.)

Problem 68. Let $X = \ell_2\text{-}\bigoplus_{n\geq 2} \ell_{2n}$. Show that then a metric ultrapower $\lim_{\mathcal{U}}(X, 0)$ of X is not reflexive.

Problem* 69. Show that for a Banach space X, a metric ultrapower $\lim_{\mathcal{U}}(X, 0)$ of X is strictly convex if and only if X is uniformly convex.

Problem* 70. Show that the class $\mathcal{H}$ilbert of all (real or complex: we fix one) Hilbert spaces is *closed under taking metric ultraproducts*, i.e., for every $(\mathcal{H}_n)_{n\in\mathbb{N}}$ with $\mathcal{H}_n \in \mathcal{H}$ilbert, there exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that

$$\lim_{\mathcal{U}}(\mathcal{H}_n, 0) \in \mathcal{H}\text{ilbert}.$$

Problem 71. For two isomorphic Banach spaces X, Y, the *Banach–Mazur distance* between X and Y is defined by

$$d_{\mathrm{BM}}(X, Y) = \inf_T (\|T\| \cdot \|T^{-1}\|).$$

Here T runs over all isomorphisms $X \to Y$ and norms above are operator norms.

For fixed $C \geq 1$, define $[\mathcal{H}\text{ilbert}]_C$ as the class of all Banach spaces X such that there exists $\mathcal{H} \in \mathcal{H}$ilbert with $d_{\mathrm{BM}}(X, \mathcal{H}) \leq C$.

Show that for every C, the class $[\mathcal{H}\text{ilbert}]_C$ is closed under taking metric ultraproducts.

Problem* 72. Complete the proof of the following proposition in the lecture.

Proposition . *Let G be a finitely generated group, and let S be a finite symmetric generating set of G. Let $(\alpha_n\colon G \curvearrowright Z_n, z_n)_{n\in\mathbb{N}}$ be a sequence of isometric actions on metric spaces $(Z_n)_n$ and base points $(z_n)_n$ ($z_n \in Z_n$ for all n). Assume that*

$$(\diamond) \qquad \sup_n \mathrm{disp}^S_{\alpha_n}(z_n) < \infty.$$

Then, the following formula

$$\alpha_{\mathcal{U}}(g)\cdot[(w_n)_n] = [(\alpha_n(g)\cdot w_n)_n]. \quad (w_n)_n \in \ell_\infty\text{-}\Pi(Z_n, z_n)$$

gives a well-defined isometric action of G on $\lim_{\mathcal{U}}(Z_n, z_n)$.

(1) Prove that for all $(w_n)_n \in \ell_\infty\text{-}\Pi(Z_n, z_n)$ and for all $s \in S$, $(\alpha_n(s)\cdot w_n)_n \in \ell_\infty\text{-}\Pi(Z_n, z_n)$.
(2) Prove that for all $(w_n)_n \in \ell_\infty\text{-}\Pi(Z_n, z_n)$ and for all $g \in G$, $(\alpha_n(g)\cdot w_n)_n \in \ell_\infty\text{-}\Pi(Z_n, z_n)$.
(3) Prove that $[(\alpha_n(g)\cdot w_n)_n]$ is well-defined, namely, it does not depend on the choice of representatives of $[(w_n)_n]$.
(4) Show that $\alpha_{\mathcal{U}}$ is an isometric action.

Problem* 73 (Warning on strong continuity)**.** We will give a *counterexample* to the strong continuity of ultraproduct actions of strongly continuous actions.

Set $\mathcal{H} = L_2(\mathbb{R})$. Define $((\pi_n, \mathcal{H}))_n$ to be the sequence of unitary representations (so, with base points being 0) as follows: For every $n \in \mathbb{N}$,

$$(\pi_n(g)\xi)(x) = \xi(x - ng) \quad g \in \mathbb{R},\ x \in \mathbb{R},\ \xi \in L_2(\mathbb{R}).$$

Consider an ultraproduct representation $\pi_{\mathcal{U}} = \lim_{\mathcal{U}}(\pi_n, 0)$ on $\mathfrak{H} = \lim_{\mathcal{U}}(\mathcal{H}, 0)$.

Let $\eta = \chi_{[0,1]} \in \mathcal{H}$, and set $\eta_{\mathcal{U}} = [(\eta)_n] \in \mathfrak{H}$. For each $g \in \mathbb{R}$, compute $\|\pi_{\mathcal{U}}(g)\eta_{\mathcal{U}} - \eta_{\mathcal{U}}\|_{\mathfrak{H}}$, and show that $\pi_{\mathcal{U}}\colon \mathbb{R} \to \mathcal{U}(\mathfrak{H})$ is *not* strongly continuous.

Problem$^\#$ 74 (Asymptotic cones and Paulin's theorem)**.** Recall from the literature the definitions of Gromov hyperbolic spaces and Gromov hyperbolic groups.

(1) Let Z be a geodesic Gromov-hyperbolic space. Show that there exists a constant δ' such that the following holds true: For all $x, y, z, w \in Z$,
$$d(x,w) + d(y,z) \le \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + \delta'.$$
(2) Let G be an (infinite) Gromov hyperbolic group, S be a finite symmetric generating set of G and $\Gamma = \mathrm{Cay}(G, S)$ with the graph metric d_Γ. Then show that for every sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n > 0$ and $\lim_{n\to\infty} t_n = \infty$ and for every $(x_n)_n$ with $x_n \in G$, the metric ultraproduct with re-scaling
$$\lim_{\mathcal{U}}(\Gamma, \frac{1}{t_n} d_\Gamma, x_n)$$
is an $\mathbb{R}$-tree, that is, 0-hyperbolic geodesic metric space.
(3) Prove the following theorem.

Theorem . *Assume that an (infinite) hyperbolic group G has property* (T)*. Then,* $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ *is a finite group.*

Note that the following generalization of Watatani's theorem is known: "*Groups with property* (T) *have the fixed point property with respect to isometric actions on* $\mathbb{R}$*-trees.*"

8. The upshot

Problem* 75. Let G be a group and X be a Banach space. Let $H \leqslant G$ be a subgroup. Let $\alpha\colon G \curvearrowright X$ be an affine isometric action of G on X. Show that if $\xi_1, \xi_2 \in X^{\alpha(H)}$, then

$$\xi_1 - \xi_2 \in X^{\rho(H)},$$

where ρ is the linear part of α.

9. The statement of Main Theorems

Problem* 76. Let R be a unital (associative) ring and $n \geq 3$. Let $G = \mathrm{St}(n, R)$ be the Steinberg group (recall the definition from the lecture). Let $\pi = (1n) \in \mathrm{Sym}(n)$ be the transposition between 1 and n in $[n]$. Show that the map

$$\phi_\pi \colon E^r_{i,j} \mapsto E^r_{\pi(i),\pi(j)}, \quad i \neq j \in [n],\ r \in R$$

gives rise to an element in $\mathrm{Aut}(G)$.

10. Metric ultraproducts and finding realizers

Problem* 77. Let G be a finitely generated group and S be a finite symmetric generating set of G. Let $\mathcal{Z}$ be a (non-empty) class of metric spaces closed under taking metric ultraproducts. Set $\mathcal{C}_{\mathcal{Z}}^{(S,1)\text{-uniform}}$ be the class of all $(S,1)$-uniform actions by isometries on $Z \in \mathcal{Z}$, i.e.,

$$\mathcal{C}_{\mathcal{Z}}^{(S,1)\text{-uniform}} = \{\alpha\colon G \curvearrowright Z : Z \in \mathcal{Z},\ \inf_{w \in Z} \mathrm{disp}^S_\alpha(w) \geq 1\}.$$

Show that then the class $\mathcal{C}_{\mathcal{Z}}^{(S,1)\text{-uniform}}$ is closed under taking metric ultraproducts, that means, for every sequence $((\alpha_n\colon G \curvearrowright Z_n, z_n))_{n\in\mathbb{N}}$ with $(\diamond)$ as in Problem 72, there exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that

$$\alpha_{\mathcal{U}} (= \lim_{\mathcal{U}} (\alpha_n, Z_n, z_n)) \in \mathcal{C}_{\mathcal{Z}}^{(S,1)\text{-uniform}}.$$

Here we forget information of the base point from $\alpha_{\mathcal{U}}$.

Problem* 78 (Gromov–Schoen argument)**.** We will show the following proposition.

Proposition . *Let G be a finitely generated group and S be a finite symmetric generating set of G. Let $\mathcal{Z}$ be a (non-empty) class of metric spaces closed under taking metric ultraproducts. Assume besides that $\mathcal{Z}$ is closed under* zooming-in. *That means, for every $(Z, d) \in \mathcal{Z}$ and for every $t \geq 1$, $(Z, td) \in \mathcal{Z}$ holds. Assume also that all $Z \in \mathcal{Z}$ is complete.*

Assume that G fails to have property $(\mathrm{F}_{\mathcal{Z}})$*. Then,*

$$\mathcal{C}_{\mathcal{Z}}^{(S,1)\text{-}uniform} \neq \emptyset.$$

We proceed to the proof. By assumption of the failure of property $(\mathrm{F}_{\mathcal{Z}})$, there exist $(Z,d)\in\mathcal{Z}$ and an isometric action $\alpha\colon G\curvearrowright Z$ such that $Z^{\alpha(G)}=\emptyset$.

(1) Prove the conclusion if $\inf_{z\in Z}\mathrm{disp}^S_\alpha(z)>0$.

(2) Let $n\in\mathbb{N}_{\geq 1}$. Show the following: There exists $z=z_n\in Z$ such that the following holds true. "*For all $w\in Z$ with $d(z,w)\leq n\mathrm{disp}^S_\alpha(z)$, $\mathrm{disp}^S_\alpha(w)\geq \frac{1}{2}\mathrm{disp}^S_\alpha(z)$ holds.*"

(Hint: Draw a contradiction by supposing that there does not exist such z.)

(3) By (1), we may assume that $\inf_{z\in Z}\mathrm{disp}^S_\alpha(z)=0$. Then for each $n\in\mathbb{N}_{\geq 1}$, show that there exists $z=z_n\in Z$ in (2) that satisfies the additional condition

$$\mathrm{disp}^S_\alpha(z_n)\leq 1.$$

(Hint. Modify the argument in (2).)

(4) Construct an $(S,1)$-uniform action from (3).

Problem* 79. Let G be a finitely generated group and S be a finite symmetric generating set of G. Let $\alpha\colon G\curvearrowright Z$ be an isometric action of G on a metric space Z that is $(S,1)$-uniform.

Show that for every $\gamma\in G$, the following new action

$$\alpha^{\mathrm{inn}(\gamma)}\colon G\curvearrowright Z;\quad \alpha^{\mathrm{inn}(\gamma)}(g)=\alpha(\gamma g\gamma^{-1})$$

is again $(S,1)$-uniform.

11. Further directions

Problem* 80. Let G be a topological group and X be a Banach space. Recall that an affine isometric action $\alpha\colon G\curvearrowright X$ is said to be continuous if for all $\xi\in X$, the orbit map of ξ,

$$G\to X;\quad g\mapsto \alpha(g)\cdot\xi$$

is continuous. This is equivalent to saying that $\rho\colon G\to O(X)$ is strongly continuous and that $b\colon G\to X$ is continuous. (Show this if you are interested in the proof.) Here ρ is the linear part of α and b is the cocycle part.

Recall the following definition from the lecture:

Definition (Displacement gap)**.** Assume that the action $\alpha\colon G\curvearrowright X$ above is continuous. Then, α is said to have a *displacement gap* if there exist $\epsilon>0$, $C\geq 0$ and a compact subset K of G such that for all $\xi\in X$,

$$\mathrm{disp}^K_\alpha(\xi)\geq \epsilon\|\bar{\xi}\|_{X/X^{\rho(G)}}-C$$

holds true. Here $\xi\mapsto\bar{\xi}$ means the natural projection from X onto $X/X^{\rho(G)}$.

Show that in the setting of the definition above, α has a displacement gap if and only if ρ (regarded as an action $\rho\colon G\curvearrowright X$) has a displacement gap with the constant C for ρ being 0.

Problem$^{\#}$ 81. Find new families of groups G in different nature to $\mathrm{E}(n, R)$ or $\mathrm{St}(n, R)$ (or other groups associated to root systems) such that we can win (Game) (or its relatives) for $(G, M_1, \ldots, M_l)$ with "good" $M_1, \ldots, M_l$ for some l.

Try to show fixed point properties for G by employing these structures.