Zadanie.
Udowodnij, że

$$\sum_{n=1}^{\infty}\left(n\cdot\log\left(\frac{2n+1}{2n-1}\right)-1\right)=\frac{1-\log 2}{2}.$$

Rozwiązanie.
Mamy

$$\sum_{n=1}^{\infty}\left(n\cdot\log\frac{2n+1}{2n-1}-1\right)=\lim_{N\to\infty}\sum_{n=1}^{N}\left(n\cdot\log\left(\frac{2n+1}{2n-1}\right)-1\right)$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}\log\left(\frac{1}{e}\cdot\left(\frac{2n+1}{2n-1}\right)^{n}\right)=\lim_{N\to\infty}\log\prod_{n=1}^{N}\left(\frac{1}{e}\cdot\left(\frac{2n+1}{2n-1}\right)^{n}\right)$$

$$=\lim_{N\to\infty}\log\left(\frac{1}{e^N}\cdot\frac{(2N+1)^N}{\prod_{n=1}^{N}(2n-1)}\right)=\lim_{N\to\infty}\log\left(\frac{1}{e^N}\cdot\frac{N!\cdot 2^N\cdot(2N+1)^N}{(2N)!}\right)$$

$$=\lim_{N\to\infty}\log\left(\frac{(2N)^{2N+\frac{1}{2}}}{(2N)!\cdot e^{2N}}\cdot\frac{N!\cdot e^N}{N^{N+\frac{1}{2}}}\cdot\frac{N^{N+\frac{1}{2}}}{(2N)^{2N+\frac{1}{2}}}\cdot 2^N\cdot(2N+1)^N\right)$$

$$=\lim_{N\to\infty}\log\left(\frac{(2N)^{2N+\frac{1}{2}}}{(2N)!\cdot e^{2N}}\cdot\frac{N!\cdot e^N}{N^{N+\frac{1}{2}}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{(2N+1)^N}{(2N)^N}\right).$$

Wiadomo, że $\lim_{n\to\infty}\frac{n!\cdot e^n}{n^{n+\frac{1}{2}}}=A>0$. Ponadto, $\lim_{n\to\infty}(1+\frac{a}{n})^n=e^a$. Zatem

$$\sum_{n=1}^{\infty}\left(n\cdot\log\left(\frac{2n+1}{2n-1}\right)-1\right)=\log\left(\frac{1}{\sqrt{2}}\cdot e^{\frac{1}{2}}\right)=\frac{1}{2}\cdot\log\left(\frac{1}{2}\cdot e\right)=\frac{1-\log 2}{2},$$

co należało udowodnić.

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