L^p-BOUNDEDNESS OF FLAG KERNELS ON HOMOGENEOUS GROUPS VIA SYMBOLIC CALCULUS

PAWEŁ GŁOWACKI

Abstract. We prove that the flag kernel singular integral operators of Nagel-Ricci-Stein on a homogeneous group are bounded on L^p, $1 < p < \infty$. The gradation associated with the kernels is the natural gradation of the underlying Lie algebra. Our main tools are the Littlewood-Paley theory and a symbolic calculus combined in the spirit of Duoandikoetxea and Rubio de Francia.

Contents

1. Introduction

Flag kernels on homogeneous groups have been introduced by Nagel-Ricci-Stein [10] in their study of quadratic CR-manifolds. They can be regarded as a generalization of Calderón-Zygmund singular kernels with singularities extending over the whole of the hyperspace $x_1 = 0$, where x_1 is the top level variable. The definition is complex, as it involves cancellation conditions for each variable separately. However, the descritption of flag kernels in terms of their Fourier transforms is much simpler and bears a striking resemblance to that of the symbols of convolution operators considered independently by the author in, e.g. [6].

In Nagel-Ricci-Stein [10] we find an L^p-boundedness theorem for the very special flag kernels where the associated gradation consists of commuting subalgebras of the underlying Lie algebra of the homogeneous group. The natural question of what happens if the gradation is the natural gradation of the homogeneous

2000 *Mathematics Subject Classification.* 43A15 (primary), 43A32, 43A85 (secondary).

Lie algebra is left open. The aim of this paper is to answer the question in the affirmative. We prove that such flag kernels give rise to bounded operators.

The smooth symbolic calculus mentioned above has been adapted to an extended class of flag kernels of small (positive and negative) orders and combined with a variant of the Littlewood-Paley theory built on a stable semigroup of measures with smooth densities very similar to the Poisson kernel on the Euclidean space (see Głowacki [5]). The strong maximal function of Christ [1] is also instrumental. The approach has been inspired by the well-known paper by Duoandicoetxea and Rubio de Francia [3]. The influence of Duoandicoetxea and Rubio de Francia [3] and, of course, Nagel-Ricci-Stein [10] is evident throughout.

A preliminary step is the L^2-boundedness of operators with flag kernels which we reproduce here for the convenience of the reader (see also Głowacki [7]). This is proved solely by means of the symbolic calculus.

After this paper had been completed, a preprint of Nagel-Ricci-Stein-Wainger [11] has been made available, where the L^p-boundedness theorem for flag kernels is proved. This comprehensive treatment of flag kernels on homogeneous groups has been announced for some time. Professor Stein has lectured a couple of times on the subject, see, e.g. [13]. The authors also use a version of Littlewood-Paley theory but otherwise the approach differs from the one presented here in many respects, the most important being our use of the symbolic calculus and partitions of unity related to a stable semigroup of measures. That is why we believe that what is presented here has an independent value and may count as a contribution to the theory.

2. Preliminaries

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a fixed Euclidean structure and $\mathfrak{g}^\star$ its dual. Let $\delta_t x = tx$, $t>0$ be a family of dilations on $\mathfrak{g}$ and let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : \delta_t x = t^{p_j} x\}, \qquad 1 \le j \le d,$$

where $1 = p_1 < p_2 < \cdots < p_d$. Denote by

$$Q_j = p_j \dim \mathfrak{g}_j$$

the homogenous dimension of $\mathfrak{g}_j$. The homogeneous dimension of $\mathfrak{g}$ is

$$Q = \sum_{j=1}^{d} Q_j.$$

We have

$$\mathfrak{g} = \bigoplus_{j=1}^{d} \mathfrak{g}_j, \qquad \mathfrak{g}^\star = \bigoplus_{j=1}^{d} \mathfrak{g}_j^\star \tag{2.1}$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \begin{cases} \mathfrak{g}_k, & \text{if } p_i + p_j = p_k, \\ \{0\}, & \text{if } p_i + p_j \notin \mathcal{P}, \end{cases}$$

where $\mathcal{P} = \{p_j : 1 \le j \le d\}$.

Let

$$x \to |x| = \sum_{j=1}^{d} \|x_j\|^{1/p_j}$$

be a homogeneous norm on $\mathfrak{g}$. Let also

$$|x|_k = \sum_{j=1}^{k} \|x_j\|^{1/p_j} = \sum_{j=1}^{k} |x_j|, \qquad 1 \leq k \leq d.$$

In particular, $|x|_1 = |x_1|$, and $|x|_d = |x|$. Another notation will be applied to $\mathfrak{g}^\star$. For $\xi \in \mathfrak{g}^\star$,

$$|\xi|_k = \sum_{j=k}^{d} \|\xi_j\|^{1/p_j} = \sum_{j=k}^{d} |\xi_j|, \qquad 1 \leq k \leq d.$$

In particular, $|\xi|_1 = |\xi|$, and $|\xi|_d = |\xi_d|$.

We shall also regard $\mathfrak{g}$ as a Lie group with the Campbell-Hausdorff multiplication

$$xy = x + y + r(x, y),$$

where $r(x, y)$ is the (finite) sum of terms of order at least 2 in the Campbell-Hausdorff series for $\mathfrak{g}$. Under this identification the homogeneous ideals

$$\mathfrak{g}^{(k)} = \bigoplus_{j=k}^{d} \mathfrak{g}_j$$

are normal subgroups.

In expressions like D^α or x^α we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k1}, \dots, \alpha_{kn_k}), \qquad n_k = \dim \mathfrak{g}_k = \dim \mathfrak{g}_k^\star,$$

are themselves multiindices with positive integer entries corresponding to the spaces $\mathfrak{g}_k$ or $\mathfrak{g}_k^\star$. The homogeneous length of α is defined by

$$|\alpha| = \sum_{k=1}^{d} |\alpha_k|, \qquad |\alpha_k| = p_k(\alpha_{k1} + \alpha_{k2} + \dots + \alpha_{kn_k}).$$

The Schwartz space of smooth functions which vanish rapidly at infinity along with their derivatives will be denoted by $\mathcal{S}(\mathfrak{g})$. This is a Fréchet space with the usual countable set of seminorms. Its dual $\mathcal{S}'(\mathfrak{g})$ is the space of tempered distributions. If $T \in \mathcal{S}'(\mathfrak{g})$ is a tempered distribution on $\mathfrak{g}$, we let

$$\langle \widetilde{T}, f \rangle = \langle T, \widetilde{f} \rangle,$$

where $\widetilde{f}(x) = f(x^{-1})$. For $t > 0$, we let

$$f_t(x) = t^{-Q} f(t^{-1}x), \qquad x \in \mathfrak{g}.$$

This extends to distributions by

$$\langle M_t, f \rangle = \langle M, f \circ \delta_t \rangle, \qquad t > 0.$$

The convolution of $f, g \in \mathcal{S}(\mathfrak{g})$ is

$$f \star g(x) = \int_{\mathfrak{g}} f(xy^{-1})g(y)\,dy.$$

where dy is Lebesgue measure which is also invariant under the group translations. The convolution is easily extended to distributions in the following way. If $T \in \mathcal{S}'(\mathcal{F})$ and $g \in \mathcal{S}(\mathfrak{g})$, then $g \star T$ is a distribution acting by

$$\langle g \star T, f\rangle = \langle T, \widetilde{g} \star f\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

If, furthermore, $S \in \mathcal{S}'(\mathfrak{g})$ has the property that $f \to f \star \widetilde{S}$ is a continuous endomorphism of $\mathcal{S}(\mathfrak{g})$, then the distribution $T \star S$ is defined by

$$\langle T \star S, f\rangle = \langle T, f \star \widetilde{S}\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

The Fourier transforms are

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} e^{-ix\xi} f(x)\,dx, \qquad f^{\vee}(\xi) = \int_{\mathfrak{g}^\star} e^{ix\xi} f(\xi)\,d\xi,$$

where the Lebesgue measures dx and $d\xi$ are normalised so that the Plancherel formula

$$\|\widehat{f}\|_2^2 = \int_{\mathfrak{g}^\star} |\widehat{f}(\xi)|^2\,d\xi = \int_{\mathfrak{g}} |f(x)|^2\,dx = \|f\|_2^2$$

holds.

Whenever we use the symbol $\star$ or refer to *convolution*, we mean the group convolution. There is one instance (proof of Proposition 4.2) where we use the vector space convolution

$$f \circ g(x) = \int_{\mathfrak{g}} f(x-y)g(y)\,dy.$$

3. Multipliers

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_d) \in \boldsymbol{R}^d$. We say that a distribution $A \in \mathcal{S}'(\mathfrak{g})$ belongs to the class $S(\mu)$, if its Fourier transform $\widehat{A}$ is a smooth function which satisfies the estimates

$$|D^\alpha \widehat{A}(\xi)| \le C_\alpha \prod_{k=1}^{d} (1 + |\xi|_k)^{\mu_k - |\alpha_k|}, \qquad \text{all } \alpha.$$

The space $S(\mu)$ is a locally convex space if endowed with the family of seminorms

$$\|A\|_{S(\mu),l} = \sup_{|\alpha|\le l} \sup_{\xi \in V^\star} \prod_{k=1}^{d} (1 + |\xi|_k)^{-\mu_k + |\alpha_k|} |D^\alpha \widehat{A}(\xi)|,$$

for $l \in \boldsymbol{N}$. Apart from the locally convex topology, one also considers *the topology of bounded convergence* in $S(\mu)$, that is the topology of uniform convergence on compact subsets of $\mathfrak{g}^\star$ of Fourier transforms and all their derivatives of sequences of elements of $S(\mu)$ bounded in the locally convex topology. Note that $\mathcal{S}(\mathfrak{g})$ is dense in $S(\mu)$ with respect to the topology of bounded convergence.

3.1. **Proposition.** *The mapping*

$$\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \ni (f,g) \mapsto f \star g \in S(\mu+\nu)$$

is continuous if the space $\mathcal{S}(\mathfrak{g})\times\mathcal{S}(\mathfrak{g})$ *is considered as a subspace of* $S(\mu)\times S(\nu)$. *It is also continuous when all the spaces are endowed with the topology of bounded convergence. By continuity, it extends to a mapping* $S(\nu) \times S(\mu) \to S(\nu+\mu)$ *which is continuous in the twofold sense.*

Proof. This follows from Corollary 5.2 of Głowacki [6]. □

3.2. **Corollary.** *Let* $A \in S(\mu)$. *Then* $f \mapsto f \star \widetilde{A}$ *is a continuous endomorphism of the Schwartz space* $\mathcal{S}(\mathfrak{g})$.

3.3. *Remark.* Let $A \in S(\mu)$, $B \in S(\nu)$. Then, by Corollary 3.2, we can define

$$\langle A \star B, f\rangle = \langle A, f \star \widetilde{B}\rangle,$$

where $A \star B \in S(\mu+\nu)$. The mapping $(A,B) \mapsto A \star B$ is the extension mapping $S(\nu) \times S(\mu) \to S(\nu+\mu)$ of Proposition 3.1.

Let

$$\mathcal{N} = \{\nu = (\nu_1, \nu_2, \ldots, \nu_d) : |\nu_k| < Q_k,\ 1 \le k \le d\}.$$

Let $\mu \in \mathcal{N}$. We say that a distribution M on $\mathfrak{g}$ belongs to the class $\mathcal{M}(\mu)$, if its Fourier transform is a locally integrable function which is smooth where $\xi_d \neq 0$ and satisfies the estimates

$$|D^\alpha \widehat{M}(\xi)| \le C_\alpha \prod_{k=1}^{d} |\xi|_k^{\mu_k - |\alpha_k|}, \qquad \xi_d \neq 0,\ \text{all } \alpha.$$

The space $\mathcal{M}(\mu)$ is a locally convex space if endowed with the family of seminorms

$$\|M\|_{\mathcal{M}(\mu),l} = \sup_{|\alpha|\le l} \sup_{\xi_d \neq 0} \prod_{k=1}^{d} |\xi|_k^{-\mu_k + |\alpha_k|} |D^\alpha \widehat{M}(\xi)|,$$

for $l \in \boldsymbol{N}$.

3.4. *Remark.* Recall that, for $M \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transform is locally integrable,

$$\widehat{M_t}(\xi) = \widehat{M}(t\xi), \qquad \xi \in \mathfrak{g}^\star,\ t > 0.$$

Therefore,

$$\|M_t\|_{\mathcal{M}(\mu),l} = t^{s(\mu)} \|M\|_{\mathcal{M}(\mu),l}, \qquad t > 0,\ l \in \boldsymbol{N},$$

where $s(\mu) = \sum_{k=1}^{d} \mu_k$.

Let $u : \mathfrak{g}^\star \to [0,1]$ be a smooth even function depending only on ξ_d and such that

$$u(\xi) = \begin{cases} 1, & \text{if } 1 \le |\xi_d| \le 2, \\ 0, & \text{if } |\xi_d| \le 1/2 \text{ or } |\xi_d| \ge 4, \end{cases}$$

and

$$\sum_{k\in\boldsymbol{Z}} u_k(\xi) = \sum_{k\in\boldsymbol{Z}} u(2^{-k}\xi) = 1, \qquad \xi_d \neq 0. \tag{3.5}$$

Let $U_k = u_k^\vee$. Note that $U_k = (U_0)_{2^{-k}}$ and $U_k \star f = f \star U_k$, for $f \in \mathcal{S}(\mathfrak{g})$. It is also clear that, for any $T \in \mathcal{S}'(\mathfrak{g})$ such that $\widehat{T}$ is locally integrable on $\mathfrak{g}^\star$,

$$T = \sum_{k \in \boldsymbol{Z}} U_k \star T = \lim_{n \to \infty} \sum_{|k| \le n} U_k \star T,$$

where the series is convergent in $\mathcal{S}'(\mathfrak{g})$.

3.6. *Remark.* We shall write

$$A(s) \approx B(s), \qquad s \in S,$$

whenever $A(s)$, $B(s)$ are quantities dependent on s and there exists a constant $C >$ such that

$$C^{-1} A(s) \le B(s) \le C A(s), \qquad s \in S.$$

3.7. **Lemma.** *If $M \in \mathcal{M}(\mu)$, then, for every k, $U_k \star M$, $U_k \star U_k \star M \in \mathcal{M}(\mu) \cap S(\mu)$, and*

$$\|U_0 \star U_0 \star M\|_{\mathcal{M}(\mu)} \approx \|U_0 \star U_0 \star M\|_{S(\mu)}.$$

Furthermore,

$$\|M\|_{\mathcal{M}(\mu),l} \approx \sup_{k \in \boldsymbol{Z}} \|U_k \star M\|_{\mathcal{M}(\mu),l}, \qquad M \in \mathcal{M}(\mu),\ l \in \boldsymbol{N}.$$

Proof. The first claim is checked directly by looking at the Fourier transforms. For the other, observe that

$$U_k \star M = \Big(U_0 \star M_{2^k}\Big)_{2^{-k}},$$

which combined with Remark 3.4 yields

$$\|U_k \star M\|_{\mathcal{M}(\mu),l} \le C \|M\|_{\mathcal{M}(\mu),l}.$$

To complete the proof it is sufficient to use the fact that the partition of unity (3.5) is uniformly locally finite. □

3.8. **Proposition.** *Let $\mu, \nu, \mu + \nu \in \mathcal{N}$. For every $M \in \mathcal{M}(\mu)$ and every $N \in \mathcal{M}(\nu)$, the sequence*

$$T_n = \sum_{|k| \le n} (U_k \star M) \star (U_k \star N)$$

is convergent in $\mathcal{S}'(\mathfrak{g})$ to an element $T \in \mathcal{M}(\mu + \nu)$, and, for every $l \in \boldsymbol{N}$, there exist $l_1, l_2 \in \boldsymbol{N}$ and a constant $C > 0$ such that

$$\|T_n\|_{\mathcal{M}(\mu+\nu),l} \le C \|M\|_{\mathcal{M}(\mu),l_1} \|N\|_{\mathcal{M}(\nu),l_2}. \tag{3.9}$$

Proof. We have

$$\|T_n\|_{\mathcal{M}(\mu+\nu),l} \approx \sup_{|k| \le n} \|(U_k \star M) \star (U_k \star N)\|_{\mathcal{M}(\mu+\nu),l},$$

and, by Proposition 3.1, Remark 3.4, and Lemma 3.7,

$$\begin{aligned}
&\|(U_k \star M) \star (U_k \star N)\|_{\mathcal{M}(\mu+\nu),l} \\
&\qquad = 2^{-s(\mu)-s(\nu)} \|(U_0 \star M_{2^k}) \star (U_0 \star N_{2^k})\|_{\mathcal{M}(\mu+\nu),l} \\
&\qquad = 2^{-s(\mu)-s(\nu)} \|(U_0 \star U_0) \star (M_{2^k} \star N_{2^k})\|_{\mathcal{M}(\mu+\nu),l} \\
&\qquad \le C_1 2^{-s(\mu)-s(\nu)} \|(U_0 \star M_{2^k}) \star (U_0 \star N_{2^k})\|_{S(\mu+\nu),l} \\
&\qquad \le C_2 2^{-s(\mu)-s(\nu)} \|U_0 \star M_{2^k}\|_{S(\mu),l_1} \|U_0 \star N_{2^k}\|_{S(\nu),l} \\
&\qquad \le C_3 2^{-s(\mu)} \|M_{2^k}\|_{\mathcal{M}(\mu),l_1} 2^{-s(\nu)} \|N_{2^k}\|_{\mathcal{M}(\nu),l_2} \\
&\qquad = C_3 \|M\|_{\mathcal{M}(\mu),l_1} \|N\|_{\mathcal{M}(\nu),l_2},
\end{aligned}$$

which gives the bound (3.9). Now, for every $\xi \in \mathfrak{g}^\star$ with $\xi_d \neq 0$, there exists n_0 such that

$$\widehat{T_n}(\xi) = \widehat{T_{n_0}}(\xi), \qquad n \ge n_0,$$

which shows that the locally integrable functions $\widehat{T_n}$ are pointwise convergent almost everywhere. By the first part of the proof,

$$|\widehat{T_n}(\xi)| \le C \prod_{j=1}^{d} |\xi|_j^{\mu_j+\nu_j},$$

where the function on the right is locally integrable, so $\widehat{T_n}$ are convergent almost everywhere to a locally integrable function $\widehat{T}$, which, by the Lebesgue dominated convergence theorem, implies $T_n \to T$ in $\mathcal{S}'(\mathfrak{g})$. □

3.10. *Remark.* If M, N, and T_n are as above, we shall write

$$M \star N = \lim_n T_n = \sum_{k \in \boldsymbol{Z}} (U_k \star M) \star (U_k \star N).$$

The following is a convenient class of test functions. Let $\mathcal{S}_0(\mathfrak{g})$ be the subspace of $f \in \mathcal{S}(\mathfrak{g})$ whose Fourier transform is disjoint with the hyperspace $\xi_d = 0$ and compact in ξ_d. The class is *total* for $\mathcal{M}(\mu)$, that is, for every $M, N \in \mathcal{M}(\mu)$,

$$\langle M, f\rangle = \langle N, f\rangle \text{ for } f \in \mathcal{S}_0(\mathfrak{g}) \implies M = N.$$

3.11. **Corollary.** *Let M, N be as above. Then $f \mapsto f \star \widetilde{N}$ is a continuous endomorphism of $\mathcal{S}_0(\mathfrak{g})$. Therefore,*

$$\langle M \star N, f\rangle = \langle M, f \star \widetilde{N}\rangle, \qquad f \in \mathcal{S}_0(\mathfrak{g}).$$

3.12. **Lemma.** *Let $|a| < Q$. Let m be locally integrable and smooth on $\mathfrak{g}^\star \setminus \{0\}$. If m satisfies*

$$|D^\alpha m(\xi)| \le C_\alpha |\xi|^{a-|\alpha|}, \qquad \xi \neq 0,$$

then the Fourier transform $k = m^\vee$ is a smooth function away from the origin and satisfies

$$|D^\alpha k(x)| \le C'_\alpha |x|^{-a-Q-|\alpha|}, \qquad x \neq 0.$$

Proof. When $a = 0$ and $|\cdot|$ is the Euclidean norm, this is Proposition 2a of Stein [12], VI.4.4. With minor corrections the same proof works in our case. □

3.13. **Proposition.** *Let $M \in \mathcal{M}_d(\nu) \cap L^2(\mathfrak{g})$. Then M satisfies the estimate*

$$|M(x)| \leq C\|M\|_{\mathcal{M}(\nu),l} \prod_{k=1}^{d} |x|_k^{-Q_k-\nu_k}, \qquad x_1 \neq 0,$$

for some $l \in \boldsymbol{N}$.

Proof. The proof is based on an argument adapted from Nagel-Ricci-Stein [10]. We proceed by induction on d. Let $M \in \mathcal{M}_d(\nu) \cap L^2(\mathfrak{g})$ and $m = \widehat{M}$. If $d = 1$, then the claim follows by Lemma 3.12. Suppose then, that $d > 1$ and the claim holds for every $1 \leq d' < d$. While taking the induction step, there is no harm in considering only multipliers $m = \widehat{M} \in L^2(\mathfrak{g})$ with compact support as long as the final estimate does not depend on the support. Let, therefore, m have compact support. Let $x \in \mathfrak{g}$, $x_1 \neq 0$. Fix $1 \leq k \leq d$ such that $|x_k| \geq d^{-1}|x|$. We split the vector space in the following way:

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'',$$

where

$$\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k-1}, \qquad \mathfrak{g}'' = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_d,$$

and write the variable x as

$$x = (x', x'') = (x_1, \ldots, x_{k-1} \mid x_k, \ldots, x_d).$$

Similarly,

$$\xi = (\xi', \xi'') \in (\mathfrak{g}^\star)' \oplus (\mathfrak{g}^\star)'' = \mathfrak{g}^\star.$$

The choice of k implies that

$$|x''| \approx |x|_j \approx |x_k|, \qquad k \leq j \leq d. \tag{3.14}$$

By definition,

$$|\xi''| = |\xi|_k.$$

The proof is carried out in three steps. First we prove that, for every $\xi_d \neq 0$,

$$|m(\xi', \cdot)^\vee(x'')| \leq C|x''|^{-Q''-N''} \prod_{j=1}^{k-1} |\xi|_j^{\nu_j}, \tag{3.15}$$

where $Q'' = \sum_{j=k}^{d} Q_j$, $N'' = \sum_{j=k}^{d} \nu_j$.

In fact, let φ be a compactly supported smooth function on $(\mathfrak{g}^\star)''$ equal to 1 in a neighbourhood of zero $|\xi''| < c$. Then,

$$\begin{aligned} m(\xi', \cdot)^\vee(x'') = &\int_{(\mathfrak{g}^\star)''} e^{ix''\xi''} \varphi(|x''|\xi'') m(\xi)\, d\xi'' \\ &+ \int_{(\mathfrak{g}^\star)''} e^{ix''\xi''} (1 - \varphi(|x''|\xi'')) m(\xi)\, d\xi'' = I_1(x'') + I_2(x''). \end{aligned}$$

The first integral is estimated by a simple change of variable:

$$\begin{aligned}|I_1(x'') &\le C_1|x''|^{-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j}\int_{(\mathfrak{g}^\star)''}|\varphi(\xi'')|\prod_{j=k}^{d}|\xi|_j^{\nu_j}\,d\xi''\\ &\le C_2|x''|^{-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j},\end{aligned}$$

We turn to $I_2(x'')$. Let us pick a unit vector $u\in\mathfrak{g}_k^\star$ such that $ux''\ge\frac{1}{2}\|x_k\|=\frac{1}{2}|x_k|^{p_k}$ and a sufficiently large integer M. Denote by ∂_u the derivative in the direction of u. Then, by integration by parts,

$$\begin{aligned}2^{-M}|x''|^{Mp_k}|I_2(x'')|\le&\left|\sum_{r=0}^{M-1}\binom{M}{r}\int e^{ix''\xi''}|x''|^{Mp_k-rp_k}\partial_u^{M-r}\varphi(|x''|\xi'')\partial_u^r m(\xi)\,d\xi''\right|\\ &+\left|\int e^{ix''\xi''}(1-\varphi(|x''|\xi'')\partial_u^M m(\xi)\,d\xi''\right|.\end{aligned}$$

The integrands in all the above integrals vanish for $|x''|\,|\xi''|<c$ so, by the change of variable,

$$\begin{aligned}2^{-M}|x''|^{Mp_k}|I_2(x'')|\le& C|x''|^{Mp_k-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j}\int_{|\xi''|\ge c}|\xi|_k^{-Mp_k}\prod_{j=k}^{d}|\xi|_j^{\nu_j}\,d\xi''\\ \le& C_1|x''|^{Mp_k-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j}\int_{(\mathfrak{g}^\star)''}(1+|\xi''|)^{-Mp_k}\prod_{j=k}^{d}|\xi_j|^{\nu_j}\,d\xi''\\ \le& C_2|x''|^{Mp_k-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j},\end{aligned}$$

where the integral on the right is convergent if $Mp_k>Q''+N''$, which finally gives

$$|I_2(x'')|\le C|x''|^{-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j}.$$

In step two we show that

$$|D_{\xi'}^\alpha m(\xi',\cdot)^\vee(x'')|\le C|x''|^{-Q''-N''}\prod_{j=1}^{k-1}|\xi|_j^{\nu_j-|\alpha_j|},\tag{3.16}$$

which is accomplished by arguing in a similar way as in step one with m replaced by $D_{\xi'}^\alpha m$.

Finally, we come to step three. By (3.16), for every x'',

$$\xi'\mapsto m_{x''}(\xi')=m(\xi',\cdot)^\vee(x'')$$

is the Fourier transform of an element in $\mathcal{M}_{k-1}(\nu_1, \ldots, \nu_{k-1})$ on $\mathfrak{g}'$ so, by the induction hypothesis,

$$|M(x)| = |m^{\vee}_{x''}(x')| \leq C|x''|^{-Q''-N''} \prod_{j=1}^{k-1} |x|_j^{-Q_j-\nu_j},$$

which combined with (3.14) completes the estimate. The argument shows also the desired dependence of the estimate on the norm $\|M\|_{\mathcal{M}(\nu),l}$, for sufficiently large l. □

More on the classes $\mathcal{M}(\mu)$ the reader will find in Głowacki [8], where they are regarded as generalised flag kernels of arbitrary order.

4. Semigroups of measures

Following Folland-Stein [4], we say that a function φ belongs to the class $\mathcal{R}(a)$, where $a > 0$, if it is smooth and

$$|D^{\alpha}\varphi(x)| \leq C_{\alpha}(1+|x|)^{-Q-a-|\alpha|}, \qquad \text{all } \alpha. \tag{4.1}$$

4.2. **Proposition.** *Let $\varphi \in \mathcal{R}(a)$ for some $0 < a < 1$ and let $\int \varphi = 0$. Then, $\widehat{\varphi} \in C^{\infty}(\mathfrak{g}^{\star} \setminus \{0\})$, and for every $b \leq a$ and every multiindex α,*

$$|D^{\alpha}\widehat{\varphi}(\xi)| \leq C_{\alpha}|\xi|^{b-|\alpha|}.$$

Proof. Let $0 \neq \xi \in \mathfrak{g}^{\star}$. Choose a unit eignevector of the dilations $u \in \mathfrak{g}$ such that $\delta_t u = t^p u$ and $u\xi \geq \frac{1}{2}|\xi|^p$, where $p \in \mathcal{P}$. Denote by ∂_u the derivative in the direction of u. For a given multiindex α, let m be an integer such that $mp \geq |\alpha|$. Let

$$\psi = (-i\partial_u)^m(-ix)^{\alpha}\varphi.$$

Since $\int \varphi = 0$, the same holds true for ψ, for any α and m as above. Moreover, $\psi \in \mathcal{R}(mp + a - |\alpha|)$. Therefore,

$$\begin{aligned}(u\xi)^m D^{\alpha}\widehat{\varphi}(\xi) = \widehat{\psi}(\xi) &= \int_{\mathfrak{g}} e^{ix\xi}\psi(x)\,dx = \int_{\mathfrak{g}} (e^{ix\xi}-1)\psi(x)\,dx \\ &= \int_{|x|\leq|\xi|^{-1}} (e^{ix\xi}-1)\psi(x)\,dx + \int_{|x|\geq|\xi|^{-1}} (e^{ix\xi}-1)\psi(x)\,dx \\ &= I_1(\xi) + I_2(\xi),\end{aligned}$$

where

$$|I_1(\xi)| \leq C_1|\xi| \int_{|x|\leq|\xi|^{-1}} |x||x|^{-Q-a-mp+|\alpha|}dx \leq C_2|\xi|^{mp+a-|\alpha|},$$

and

$$|I_2(\xi)| \leq 2\int_{|x|\geq|\xi|^{-1}} |x|^{-Q-a-mp+|\alpha|}\,dx \leq C_2|\xi|^{mp+a-|\alpha|},$$

which implies

$$|D^{\alpha}\widehat{\varphi}(\xi)| \leq C_{\alpha}|\xi|^{a-|\alpha|}.$$

Thus we get our estimate for $b = a$.

The proof will be completed once we show that for every α and every $N \in \boldsymbol{N}$, there exists a constant $C_N > 0$ such that

$$|D^\alpha \widehat{\varphi}(\xi)| \leq C_N |\xi|^{-N}, \qquad |\xi| \geq 1.$$

Fix α and let $\psi = (-ix)^\alpha \varphi(x)$. We will take advantage of the Taylor expansion formula

$$\psi(x-y) = \sum_{|\beta|<N} \frac{D^\beta \psi(x)}{\beta!}(-y)^\beta + R_N(x,y),$$

where

$$|R_N(x,y)| \leq K_N (1+|x|)^{-Q-a-N+|\alpha|}(1+|y|)^{Q+a+N-|\alpha|}|y|^N, \tag{4.3}$$

where $K_N > 0$ is a constant and $N \geq |\alpha|$.

Denote by $\circ$ the vector space convolution on $\mathfrak{g}$. Let $f \in \mathcal{S}(\mathfrak{g})$ be such that $\widehat{f}$ is supported where $1/2 \leq |\xi| \leq 4$ and equal to one where $1 \leq |\xi| \leq 2$. By definition,

$$\int x^\beta f(x)\, dx = 0,$$

for all β, hence,

$$f_t \circ \psi(x) = \int f_t(y)\psi(x-y)\, dy = \int f_t(y) R_N(x,y)\, dy,$$

where $f_t(x) = t^{-Q} f(t^{-1}x)$. Therefore, by (4.3),

$$|f_t \circ \psi(x)| \leq L_N t^N (1+|x|)^{-Q-a-N-|\alpha|}, \qquad 0 < t \leq 1,$$

which implies

$$|\widehat{f}(t\xi)\widehat{\psi}(\xi)| \leq C_N t^N, \qquad 0 < t \leq 1,$$

and, consequently, by letting $t = |\xi|^{-1}$,

$$|D^\alpha \widehat{\varphi}(\xi)| = |\widehat{\psi}(\xi)| \leq C_N |\xi|^{-N} \qquad |\xi| \geq 1,$$

which completes the proof. □

Let

$$\langle P, f\rangle = \lim_{\varepsilon\to} \int_{|x|\geq\varepsilon} \Big(f(0) - f(x)\Big)\frac{dx}{|x|^{Q+1}}, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

The distribution $-P$ is a *generalised laplacian* (see Duflo [2], Section 2) and, therefore, a generating functional of a continuous semigroup of probability measures μ_t (Hunt [9]). In other words,

$$\mu_t \star \mu_s = \mu_{t+s}, \qquad t,s > 0,$$

and

$$\lim_{t\to 0}\langle \mu_t, f\rangle = f(0), \qquad f \in \mathcal{S}(\mathfrak{g}),$$

as well as

$$\frac{d}{dt}\Big|_{t=0}\langle \mu_t, f\rangle = -\langle P, f\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

(See Duflo [2], Proposition 4 or Hunt [9]) The operator $\mathbf{P}f = f \star P$ is positive and essentially selfadjoint with $\mathcal{S}(\mathfrak{g})$ for its core domain. The densities h_t belong to the domain of $\mathbf{P}$. It is also an infinitesimal generator of a strongly continuous semigroup of contractions

$$T_t = f \star h_t, \qquad t > 0,$$

on the Hilbert space $L^2(\mathfrak{g})$ (see Duflo [2], Example 4, p 247).

By Theorem 2.3 of Głowacki [5], the measures μ_t have smooth desities h_t, where $h_1 \in \mathcal{R}(1)$. By the definition of the distribution P,

$$\mathbf{P}(f_t) = t^{-1}(\mathbf{P}f)_t, \tag{4.4}$$

which implies

$$h_t(x) = (h_1)_t(x) = t^{-Q}h_1(t^{-1}x), \qquad x \in \mathfrak{g},\ t > 0.$$

For $0 < a < 1$, let $\mathbf{P}^a$ be the fractional power of $\mathbf{P}$. By Yosida [14], Theorem 2, IX.11,

$$\mathbf{P}^a f = \frac{1}{\Gamma(-a)}\int_0^\infty t^{-1-a}(I - e^{-t\mathbf{P}})f\,dt = \frac{1}{\Gamma(1-a)}\int_0^\infty t^{-a}\mathbf{P}(f \star h_t)\,dt. \tag{4.5}$$

4.6. **Proposition.** *For every* $0 < a < 1$*,*

$$\mathbf{P}^a h_1 \in \mathcal{R}(a) \qquad \text{and} \qquad \int_{\mathfrak{g}} \mathbf{P}^a h_1(x)\,dx = 0.$$

Proof. By (4.5),

$$\mathbf{P}^a h_1(x) = \frac{1}{\Gamma(1-a)}\int_0^\infty t^{-a}\mathbf{P}h_{t+1}(x)\,dt,$$

whence

$$\begin{aligned}
|D^\alpha \mathbf{P}^a h_1(x)| &\le \frac{C_\alpha}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}\,dt}{(t+1+|x|)^{Q+1+|\alpha|}} \\
&\le C'_\alpha \int_0^\infty \frac{t^{-a}\,dt}{(\frac{t}{1+|x|}+1)^{Q+1+|\alpha|}} \cdot (1+|x|)^{-Q-1-|\alpha|} \\
&\le C''_\alpha \int_0^\infty \frac{t^{-a}\,dt}{(t+1)^{Q+1+|\alpha|}} \cdot (1+|x|)^{-Q-a-|\alpha|} \\
&\le C'''_\alpha (1+|x|)^{-Q-a-|\alpha|},
\end{aligned}$$

as required.

Now, for every $t > 0$,

$$\int_{\mathfrak{g}} h_t\,dx = 1.$$

Therefore,

$$\int_{\mathfrak{g}} \mathbf{P}h_t\,dx = -\frac{d}{dt}\int_{\mathfrak{g}} h_t\,dx = 0, \qquad t > 0.$$

which combined with (4.5) gives the second part of the assertion. □

5. Littlewood-Paley theory

From now on we fix the function

$$\varphi = \mathbf{P}^{1/2} h_{1/2} = \sqrt{2}\Big(\mathbf{P}^{1/2} h_1\Big)_{1/2}.$$

Note that

$$\varphi \star \varphi = \mathbf{P}^{1/2} h_{1/2} \star \mathbf{P}^{1/2} h_{1/2} = T_{1/2}\mathbf{P}^{1/2}(\mathbf{P}^{1/2} h_{1/2}) = \mathbf{P}(T_{1/2} h_{1/2}) = \mathbf{P} h_1,$$

hence, by (4.4),

$$\varphi_t \star \varphi_t = (\varphi \star \varphi)_t = (\mathbf{P} h_1)_t = t \mathbf{P} h_t.$$

5.1. *Remark.* By Propositions 4.2 and 4.6, φ is a smooth function, and

$$|D^\alpha \varphi(x)| \le C_\alpha (1+|x|)^{-Q-1/2-|\alpha|} \tag{5.2}$$

as well as

$$|D^\alpha \widehat{\varphi}(\xi)| \le C_\alpha |\xi|^b |\xi|^{-|\alpha|}, \qquad \xi \ne 0,$$

for $|b| \le 1/2$.

5.3. **Lemma.** *We have*

$$f = \int_0^\infty f \star \varphi_t \star \varphi_t \,\frac{dt}{t}, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

where the integral is convergent in the $L^2(\mathfrak{g})$-norm. In particular,

$$\int_0^\infty \langle f \star \varphi_t, g \star \varphi_t\rangle = \langle f, g\rangle, \qquad f, g \in L^2(\mathfrak{g}).$$

Proof. By the semigroup properties,

$$-\frac{d}{dt} f \star h_t = f \star P h_t = \frac{1}{t} f \star (\varphi_t \star \varphi_t),$$

whence

$$\int_\varepsilon^M f \star \varphi_t \star \varphi_t \frac{dt}{t} = f \star h_\varepsilon - f \star h_M.$$

Now, if $\varepsilon \to 0$ and $M \to \infty$, the expression on the right-hand side tends to f in $L^2(\mathfrak{g})$. □

5.4. **Proposition.** *Let*

$$g_\varphi(f)(x) = \left(\int_0^\infty |f \star \varphi_t(x)|^2 \,\frac{dt}{t}\right)^{1/2},$$

be the Littlewood-Paley square function operator. Then,

$$\|g_\varphi(f)\|_2 = \|f\|_2, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. Let $f \in \mathcal{S}(\mathfrak{g})$. By Lemma 5.3,

$$\begin{aligned} \|f\|_2^2 = \langle f, \bar f\rangle = \int_0^\infty \langle f \star \varphi_t \star \varphi_t, \bar f\rangle \frac{dt}{t} &= \int_0^\infty \int_{\mathfrak{g}} |f \star \varphi_t(x)|^2 dx \,\frac{dt}{t} \\ &= \int_{\mathfrak{g}} \int_0^\infty |f \star \varphi_t(x)|^2 \,\frac{dt}{t}\, dx = \|g_\varphi\|_2^2. \end{aligned}$$

□

Let $T=(t_1,\dots,t_d)\in \boldsymbol{R}_+^d$. We shall regard $\boldsymbol{R}_+^d$ as a product of copies of the multiplicative group $\boldsymbol{R}^+$. We shall write

$$T^a=(t_1^a,\dots,t_d^a),\qquad TS=(t_1s_1,\dots,t_ds_d),\qquad \frac{dT}{T}=\frac{dt_1\dots dt_d}{t_1\dots t_d},\qquad a\in\boldsymbol{R}.$$

Let φ_k be the counterpart of φ for $\mathfrak{g}$ replaced by $\mathfrak{g}^{(k)}$, $1\le k\le d$. Let

$$\Phi_k=\delta_k\otimes\varphi_k,$$

where δ_k stands for the Dirac delta at $0\in\oplus_{j=1}^{k-1}\mathfrak{g}_j$.

5.5. **Corollary.** *If*

$$g_{\Phi_k}(f)(x)=\left(\int_0^\infty |f\star(\Phi_k)_t(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

then

$$\|g_{\Phi_k}f\|_2=\|f\|_2,\qquad f\in\mathcal{S}(\mathfrak{g}).$$

Proof. This is a direct consequence of Proposition 5.4. □

Let

$$\Phi=\Phi_1\star\Phi_2\star\dots\star\Phi_d,$$

and

$$\Phi_T=(\Phi_1)_{t_1}\star\dots\star(\Phi_d)_{t_d},\qquad T\in\boldsymbol{R}_+^d.$$

5.6. **Corollary.** *For every T,*

$$\Phi_T\in\mathcal{S}(\mathfrak{g}),$$

and fo every $\nu\in[-1/2,1/2]^d$,

$$\|\Phi_T\|_{\mathcal{M}(\nu),l}\le C_l\prod_{k=1}^d t_k^{\nu_k}.$$

Proof. That $\Phi_T\in\mathcal{S}(\mathfrak{g})$ is a simple exercise. By Remark 5.1,

$$(\Phi_k)_{t_k}\in\bigcap_{|b|\le 1/2}\mathcal{M}(0,\dots 0,b,0,\dots,0),$$

where the only nonzero term stands on the k-th position, and

$$\|(\Phi_k)_{t_k}\|_{\mathcal{M}(0,\dots0,b,0,\dots,0),l}\le c_l t_k^b.$$

Therefore the assertion follows by Proposition 3.8. □

5.7. **Corollary.** *We have*

$$\langle f,g\rangle=\int_{\boldsymbol{R}_+^d}\langle f\star\Phi_T,g\star\Phi_T\rangle\,\frac{dT}{T},\qquad f,g\in L^2(\mathfrak{g}).$$

Proof. This follows from Lemma 5.3 by iteration. □

5.8. **Proposition.** *The Littlewood-Paley square function operator*

$$G_\Phi(f)(x) = \left(\int_{\boldsymbol{R}_+^d} |f \star \Phi_T(x)|^2 \, \frac{dT}{T} \right)^{1/2},$$

is of type (p,p)*, for every* $1 < p < \infty$*. That is, there exists a constant* C *dependent on* p *such that*

$$\|G_\Phi(f)\|_p \leq C\|f\|_p, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. The proof is implicitly contained in Folland-Stein [4] (see Theorem 6.20.b and Theorem 7.7) so we dispense ourselves with presenting all details.

We start with defining some Hilbert spaces and operators. Let $X_0 = \boldsymbol{C}$ and

$$X_k = L^2(\boldsymbol{R}_+^k, \frac{dT}{T}), \qquad 1 \leq k \leq d.$$

Let $W_k : L^2(\mathfrak{g}, X_{k-1}) \to L^2(\mathfrak{g}, X_k)$ be the operator

$$W_k f(x)(T, t_k) = f \star (\Phi_k)_{t_k}(x)(T) = \int_{\mathfrak{g}^{(k)}} (\varphi_k)_{t_k}(y) f(xy)(T) \, dy,$$

where $T = (t_1, \ldots, t_{k-1})$. Note that W_k acts only on the $(x_k, \ldots, x_d)$-variable. We can also write

$$W_k f(x) = \int_{\mathfrak{g}^{(k)}} w_k(y) f(xy) \, dy,$$

where, for every $y \in \mathfrak{g}^{(k)}$ and every $m \in X_{k-1}$,

$$w_k(y) : X_{k-1} \to X_k, \qquad w_k(y)m(T, t_k) = (\varphi_k)_{t_k}(y)m(T).$$

We claim that W_k is a bounded operator, even an isometry. In fact, by the definition of Φ_k and Corollary 5.5,

$$\begin{aligned}
\|W_k f\|^2_{L^2(\mathfrak{g}, X_k)} &= \int_{\mathfrak{g}} \|W_k f(x)\|^2_{X_k} \, dx \\
&= \int_{\mathfrak{g}} dx \int_0^\infty \frac{dt}{t} \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \int_{\mathfrak{g}^{(k)}} |(\varphi_k)_t(y) f(xy)(T)|^2 \, dy \\
&= \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} \int_{\mathfrak{g}} \int_{\mathfrak{g}^{(k)}} |(\varphi_k)_t(y) f(xy)(T)|^2 \, dy dx, \\
&= \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} \langle f_T \star (\Phi_k)_t, f_T \star (\Phi_k)_t \rangle \\
&= \int_{\boldsymbol{R}_+^{k-1}} \|g_{\Phi_k}(f_T)\|_2^2 \frac{dT}{T} = \int_{\boldsymbol{R}_+^{k-1}} \|f_T\|_2^2 \frac{dT}{T} = \|f\|^2_{L^2(\mathfrak{g}, X_{k-1})},
\end{aligned}$$

where $f_T(x) = f(x)(T)$.

Another property of the kernel w_k of W_k that is needed is the following. For every α

$$\|D^\alpha w_k(x)\|_{(X_{k-1}, X_k)} \leq C_\alpha |x|_k^{-Q^{(k)} - |\alpha|}, \tag{5.9}$$

where $Q^{(k)} = Q_k + Q_{k+1} + \cdots + Q_d$. This follows readily from (5.2) specialized to φ_k:

$$|D^\alpha \varphi_k(x)| \le C_\alpha (1 + |x|_k)^{-Q^{(k)}-1/2-|\alpha|}.$$

As a bounded operator from $L^2(\mathfrak{g}, X_{k-1})$ to $L^2(\mathfrak{g}, X_k)$ satisfying (5.9) is W_k a vector-valued kernel of type 0, and, by Theorem 6.20.b of Folland-Stein [4], maps $L^p(\mathfrak{g}, X_{k-1})$ into $L^p(\mathfrak{g}, X_k)$ boundedly for every $1 < p < \infty$.

This implies our assertion. In fact,

$$G_\Phi(f)(x) = \|W_d W_{d-1} \dots W_1 f(x)\|_{X_d},$$

and therefore

$$\|G_\Phi(f)\|_{L^p(\mathfrak{g})} = \|W_d W_{d-1} \dots W_1 f\|_{L^p(\mathfrak{g}, X_d)} \le C\|f\|_{L^p(\mathfrak{g}, X_0)} = C\|f\|_p.$$

□

A word of comment on the symbol Φ_T would be appropriate here. The notation may suggest that the functions Φ_T are dilates of a single function. They are not, but they have estimates of this form, which is our justification. In the next section we are going to use the same notation for the "real" dilates of a function. We hope the reader will not get confused.

6. The strong maximal function

Let

$$Tx \mapsto (t_1 x_1, t_2 x_2, \dots, t_d x_d), \qquad x \in \mathfrak{g}, T \in \boldsymbol{R}_+^d.$$

For a function F on $\mathfrak{g}$ and a $T \in \boldsymbol{R}_+^d$, let

$$F_T(x) = t_1^{-Q_1} t_2^{-Q_2} \dots t_d^{-Q_d} F(T^{-1}x).$$

Let B_j be the unit ball in $\mathfrak{g}_j$ and let $D = B_1 \times \cdots \times B_d$. Let $|D|$ be the Lebesgue measure of D. *The strong maximal function* on $\mathfrak{g}$ is defined by

$$\mathbf{M}f(x) = \sup_{T \in \boldsymbol{R}_+^d} \int_D |f(x(Ty)^{-1})| \, dy = \sup_T |f| \star (\chi_D)_T(x). \tag{6.1}$$

A theorem of Michael Christ asserts that, for every $1 < p < \infty$, there exists a constant $C > 0$ such that

$$\|\mathbf{M}f\|_p \le C\|f\|_p, \qquad f \in L^p(\mathfrak{g}),$$

that is, $\mathbf{M}$ is of (p, p) type (see Christ [1]). Actually, Christ considers a slightly different but obviously equivalent maximal function, where χ_D is replaced with χ_B, B being a unit ball in $\mathfrak{g}$. .

We shall need the following corollary to the Christ theorem. Let

$$\gamma(r) = \min\{r, r^{-1}\}, \qquad r > 0.$$

6.2. **Corollary.** *Let*

$$F(x) = \prod_{j=1}^d \gamma(|x_j|)^a |x_j|^{-Q_j}, \qquad x \ne 0,$$

for some $a > 0$. Then the maximal function

$$M_F f(x) = \sup_{T \in \boldsymbol{R}_+^d} |f| \star F_T(x)$$

is of (p,p) type for $1 < p < \infty$.

Proof. The function F is radially decreasing in each variable so

$$F(x) = \sup_{h_{\mathcal{R}} \leq F} h_{\mathcal{R}}(x),$$

and

$$\int F(x)\, dx = \sup_{h_{\mathcal{R}} \leq F} \int h_{\mathcal{R}}(x)\, dx,$$

where

$$h_{\mathcal{R}} = \sum_{R \in \mathcal{R}} c_R \chi_D(R^{-1}x), \qquad R = (r_1, r_2, \dots, r_d) \in \boldsymbol{R}_+^d,$$

and $\mathcal{R} \subset \boldsymbol{R}_+^d$ is a finite set. For $f \geq 0$, we have

$$\begin{aligned} f \star (h_{\mathcal{R}})_T(x) &= \sum_R c_R r_1^{Q_1} r_2^{Q_2} \dots r_d^{Q_d} f \star (\chi_D)_{RT}(x) \\ &\leq \Big(\frac{1}{|D|} \sum_R c_R r_1^{Q_1} r_2^{Q_2} \dots r_d^{Q_d} |D| \Big) \mathbf{M} f(x) \\ &\leq \frac{\|h_{\mathcal{R}}\|_1}{|D|} \mathbf{M} f(x) \end{aligned}$$

and therefore

$$M_F f(x) \leq \frac{\|F\|_1}{|D|} \mathbf{M} f(x),$$

which completes the proof. □

7. Flag kernels

In the context of this paper it is natural to work with a description of flag kernels given in terms of flag multipliers (see Nagel-Ricci-Stein [10], Theorem 2.3.9). We say that a tempered distribution K on $\mathfrak{g}$ is a *flag kernel* if its Fourier transform $\widehat{K}$ is a smooth function where $\xi_d \neq 0$ and satisfies the following estimates

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k=1}^d |\xi|_k^{-|\alpha_k|}, \qquad \xi_d \neq 0, \text{ all } \alpha.$$

Thus, K is a flag kernel if and only if $K \in \mathcal{M}(\mathbf{0})$, where $\mathbf{0} = (0, 0, \dots, 0)$.

7.1. **Lemma.** *The class $\mathcal{S}_0(\mathfrak{g})$, as defined in Section 3 before Corollary 3.11, is a dense subspace of $L^p(\mathfrak{g})$, for $1 < p < \infty$.*

Proof. Let $1/q + 1/p = 1$, and let $g \in L^q(\mathfrak{g})$ be such that

$$\int_{\mathfrak{g}} f(x) g(x)\, dx = 0, \qquad f \in \mathcal{S}_0(\mathfrak{g}).$$

We are going to show that $g = 0$, which implies the required density. For $\varphi \in \mathcal{S}(\mathfrak{g}')$, where $\mathfrak{g}' = \mathfrak{g}_1 \oplus \dots \mathfrak{g}_{d-1}$, let

$$g_1(x_d) = \int_{\mathfrak{g}'} \varphi(x')g(x', x_d)\, dx'.$$

Then, $g_1 \in L^q(\mathfrak{g}_d)$ and, by hypothesis, $\widehat{g_1} \in \mathcal{S}'(\mathfrak{g}_g)$ is supported at the origin. Therefore, g_1 is also a polynomial so it must be zero. Since $\varphi \in \mathcal{S}(\mathfrak{g}')$ was arbitrary, g itself must be zero. □

7.2. **Lemma.** *If K is a flag kernel and $f \in \mathcal{S}_0(\mathfrak{g})$, then $f \star K \in \mathcal{S}_0(\mathfrak{g})$. In other words, $\mathcal{S}_0(\mathfrak{g})$ is invariant under convolutions with flag kernels.*

Proof. This is a special case of Corollary 3.11. □

7.3. **Proposition** (Theorem 2.5 of [7])**.** *Let K be a flag kernel on $\mathfrak{g}$. The convolution operator $f \mapsto f \star K$ defined initially on $\mathcal{S}_0(\mathfrak{g})$ extends uniquely to a bounded operator $\boldsymbol{K}$ on $L^2(\mathfrak{g})$ and there exists a constant $C > 0$ and an integer l such that*

$$\|\boldsymbol{K}\| \le C\|K\|_{\mathcal{M}(\mathbf{0}),l}.$$

Proof. By Theorem 5.5 of Głowacki [6], there exists a constant $C > 0$ and an integer $l \in \boldsymbol{N}$ such that

$$\|f \star T\|_2 \le C\|T\|_{S(\mathbf{0}),l}\|f\|_2, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{7.4}$$

for $T \in S(\mathbf{0})$. By the definition of the distributions U_k (see Secton 3) and the Plancherel theorem,

$$\|f\|_2^2 \approx \sum_{k\in\boldsymbol{Z}} \|U_k \star f\|_2^2 \approx \sum_{k\in\boldsymbol{Z}} \|U_k \star U_k \star f\|_2^2, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{7.5}$$

where the symbol $\approx$ has been defined in Remark 3.6. Recall also that $U_k = (U_0)_{2^{-k}}$. Therefore, by Remark 3.4, Lemma 3.7, and (7.4),

$$\begin{aligned}\|U_k \star U_k \star f \star K\|_2^2 &= 2^{kQ}\|(U_0 \star f_{2^k}) \star (U_0 \star K_{2^k})\|_2^2\\ &\le 2^{kQ}\|U_0 \star K_{2^k}\|_{S(\mathbf{0}),l}^2\|U_0 \star f_{2^k}\|_2^2\\ &\approx 2^{kQ}\|U_0 K_{2^k}\|_{\mathcal{M}(\mathbf{0}),l}^2\|U_0 \star f_{2^k}\|_2^2\\ &\le C\|K\|_{\mathcal{M}(\mathbf{0}),l}^2\|f \star U_k\|_2^2,\end{aligned}$$

which combined with (7.5) gives the required estimate. □

7.6. *Remark.* The convolution of flag kernels can be understood in two equivalent ways: in terms of Proposition 3.8, Remark 3.10, and Corollary 3.11, or as a composition of two convolvers on $L^2(\mathfrak{g})$ (Proposition 7.3).

7.7. **Proposition** (Theorem 2.2 of [7])**.** *If K_1, K_2 are flag kernels on $\mathfrak{g}$, then $K_1 \star K_2$ is also a flag kernel.*

Proof. This is a corollary to Proposition 3.8. □

We keep the notation established in previous sections.

7.8. **Lemma.** *Let*

$$K_{T,S} = \widetilde{\Phi_{TS}} \star \widetilde{K} \star \Phi_T, \qquad T, S \in \boldsymbol{R}_+^d.$$

For every $T, S \in \boldsymbol{R}_+^d$, $K_{T,S}$ is an integrable function such that

$$|K_{T,S}(x)| \leq C\gamma(S)^{1/4} \prod_{k=1}^{d} \gamma(|t_k x|_k)^{1/4} |x|_k^{-Q_k}$$

where

$$\gamma(S) = \gamma(s_1)\gamma(s_2)\cdots\gamma(s_d), \qquad \gamma(s) = \min\{s, s^{-1}\}.$$

Proof. By Corollary 5.6, for every $\mu, \nu, \mu+\nu \in [-1/2, 1/2]^d$, there exist constants C_α such that, for all $\xi \in \mathfrak{g}^\star$ with $\xi_d \neq 0$ and all $s_k > 0, t_k > 0$,

$$|D^\alpha \widetilde{\Phi_{TS}}^{\wedge}(\xi)| \leq C_\alpha \prod_{k=1}^{d} (t_k s_k |\xi|_k)^{\mu_k} |\xi|_k^{-|\alpha_k|},$$

$$|D^\alpha \widehat{\Phi_T}(\xi)| \leq C_\alpha \prod_{k=1}^{d} (t_k |\xi|_k)^{\nu_k} |\xi|_k^{-|\alpha_k|},$$

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k=1}^{d} |\xi|_k^{-|\alpha_k|},$$

which, by Proposition 3.8, yields

$$|D^\alpha \widehat{K_{T,S}}(\xi)| \leq C'_\alpha \prod_{k=1}^{d} s_k^{\mu_k} (t_k |\xi|_k)^{\mu_k+\nu_k} |\xi|_k^{-|\alpha_k|}, \qquad \xi_d \neq 0, t_k > 0, s_k > 0.$$

Note also that, by Propositions 5.6 and 7.3, $K_{T,S} \in L^2(\mathfrak{g})$. Hence, by Proposition 3.13,

$$|K_{T,S}(x)| \leq C \prod_{k=1}^{d} s_k^{\mu_k} t_k^{\mu_k+\nu_k} |x|_k^{-\mu_k-\nu_k} |x|_k^{-Q_k}.$$

By choosing appropriately $\mu_k = \pm 1/4$ and $\nu_k = 0, \pm 1/2$ depending on whether s_k and $t_k^{-1}|x|_k$ are smaller or greater than 1, we get our assertion. □

7.9. **Corollary.** *For a given $S \in \boldsymbol{R}_+^d$, the maximal operator*

$$K_S^\star f(x) = \sup_T |f| \star |\widetilde{K_{T,S}}|(x)$$

is of type (p,p) with

$$\|K_S^\star f\|_p \leq C\gamma(S)^{1/4} \|f\|_p, \qquad f \in L^p(\mathfrak{g}).$$

Proof. By Lemma 7.8,

$$|K_{T,S}(x)| \leq C\gamma(S)^{1/4} F_T(x),$$

where

$$F(x) = \prod_{k=1}^{d} \gamma(|x|_k) |x|_k^{-Q_k}.$$

Thus the assertion follows by Corollary 6.2. □

7.10. **Lemma.** *Let* $K \in L^1(\mathfrak{g})$, $g \in L^2(\mathfrak{g})$. *Then,*

$$|g \star K(x)|^2 \le \|K\|_1 \, |g|^2 \star |K|(x), \qquad x \in \mathfrak{g}.$$

Proof. In fact,

$$\begin{aligned}|g \star K(x)|^2 &\le \left(\int_{\mathfrak{g}} |g(xy^{-1})| \cdot |K(y)|^{1/2} \cdot |K(y)|^{1/2} \, dy \right)^2 \\ &\le \int_{\mathfrak{g}} |g|^2(xy^{-1}) \cdot |K|(y) \, dy \cdot \int_{\mathfrak{g}} |K(y)| \, dy \\ &\le \|K\|_1 \, |g|^2 \star |K|(x),\end{aligned}$$

for every $x \in \mathfrak{g}$. □

We turn to the main result of this paper. The reader may wish to compare the proof we give with that of Theorem B and the preceding lemma of Duoandicoetxea-Rubio de Francia [3].

7.11. **Theorem.** *Let* K *be a flag kernel on* $\mathfrak{g}$. *For every* $1 < p < \infty$, *the singular integral operator*

$$f \to f \star K, \qquad f \in \mathcal{S}_0(\mathfrak{g}),$$

extends uniquely to a bounded operator $\boldsymbol{K}$ *on* $L^p(\mathfrak{g})$, *and there exists a constant* $C > 0$ *and an integer* l *such that*

$$\|\boldsymbol{K}\| \le C \|K\|_{\mathcal{M}(\mathbf{0}),l}.$$

Proof. By Lemma 7.1, we may choose $\mathcal{S}_0(\mathfrak{g})$ as our space of test functions. Let $f, g \in \mathcal{S}_0(\mathfrak{g})$. By Corollary 5.7, Proposition 7.7, and Lemma 7.2,

$$\begin{aligned}\langle f \star K, g \rangle = \langle f, g \star \widetilde{K} \rangle &= \int_{\boldsymbol{R}_+^d} \langle f \star \Phi_T, g \star \widetilde{K} \star \Phi_T \rangle \frac{dT}{T} \\ &= \int_{\boldsymbol{R}_+^d} \int_{\boldsymbol{R}_+^d} \langle f \star \Phi_T, g \star \Phi_{TS} \star \widetilde{\Phi_{TS}} \star \widetilde{K} \star \Phi_T \rangle \frac{dS}{S} \frac{dT}{T} \\ &= \int_{\boldsymbol{R}_+^d} \int_{\boldsymbol{R}_+^d} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T} \frac{dS}{S},\end{aligned}$$

where

$$f_T = f \star \Phi_T, \qquad g_{TS} = g \star \Phi_{TS}, \qquad K_{T,S} = \widetilde{\Phi_{TS}} \star \widetilde{K} \star \Phi_T.$$

We are going to estimate

$$L_S(f,g) = \int_{\boldsymbol{R}_+^d} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T},$$

for a given S. Recall that, by Proposition 5.8, the square function G_Φ of Section 5 is of type (p,p), for every $1<p<\infty$. Therefore, for $1<p<2$ and $f,g\in\mathcal{S}_0(\mathfrak{g})$,

$$\begin{aligned}|\langle L_S f,g\rangle| &\le \int_{\mathfrak{g}}\left(\int_{\boldsymbol{R}^d_+}|f_T(x)|^2\frac{dT}{T}\right)^{1/2}\left(\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(x)|^2\,\frac{dT}{T}\right)^{1/2}dx\\ &\le C_1\|G_\Phi(f)\|_p\left(\int_{\mathfrak{g}}\left(\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(x)|^2\,\frac{dT}{T}\right)^{q/2}dx\right)^{1/q}\\ &= C_2\|f\|_p\cdot\left\|\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(\cdot)|^2\,\frac{dT}{T}\right\|_{q/2}^{1/2},\end{aligned}$$

where $1/p+1/q=1$. Note that $q>2$. Thus, there exists a nonnegative function u with $\|u\|_r=1$, where $2/q+1/r=1$, such that

$$\left\|\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(\cdot)|^2\,\frac{dT}{T}\right\|_{q/2}=\int_{\mathfrak{g}}\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(x)|^2\,\frac{dT}{T}\,u(x)\,dx.$$

Therefore, by Corollary 7.9, Lemma 7.8, and Lemma 7.10,

$$\begin{aligned}\left\|\int_{\boldsymbol{R}^d_+}|g_{TS}\star K_{T,S}(\cdot)|^2\,\frac{dT}{T}\right\|_{q/2} &\le C\gamma(S)^{1/4}\int_{\boldsymbol{R}^d_+}\int_{\mathfrak{g}}|g_{TS}|^2\star|K_{T,S}|(x)\,u(x)\,dx\frac{dT}{T}\\ &\le C\gamma(S)^{1/4}\int_{\mathfrak{g}}\int_{\boldsymbol{R}^d_+}|g_{TS}(x)|^2\,\frac{dT}{T}\cdot K^\star_S u(x)\,dx\\ &\le C_1\gamma(S)^{1/4}\|G_\Phi(g)\|_q^2\cdot\|K^\star_S u\|_r\le C_2\gamma(S)^{1/2}\|g\|_q^2, \qquad (7.12)\end{aligned}$$

whence

$$|L_S(f,g)|\le C\gamma(S)^{1/4}\|f\|_p\|g\|_q.. \qquad (7.13)$$

Finally,

$$|\langle f\star K,g\rangle|\le C\left(\int_{\boldsymbol{R}^d_+}\gamma(S)^{1/4}\,\frac{dS}{S}\right)\|f\|_p\|g\|_q=C_1\|f\|_p\|g\|_q,$$

which proves our case for $1<p<2$. The result for $2<p<\infty$ follows by duality. The case $p=2$ has been already established in Lemma 7.3. □

7.14. **Corollary.** *Let $1<p<\infty$. Let K be a flag kernel on $\mathfrak{g}$. For each $n\in\boldsymbol{N}$, let*

$$K_n=\sum_{|k|\le n}U_k\star K,\qquad \boldsymbol{K}_n f=f\star K_n.$$

Then, for every $f\in L^p(\mathfrak{g})$,

$$\|\boldsymbol{K}_n f-\boldsymbol{K}f\|_p\to 0,\qquad n\to\infty.$$

Proof. By Lemma 3.7,

$$\|K_n\|_{\mathcal{M}(\mathbf{0}),l}\le C\max_{|k|\le n}\|U_k\star K\|_{\mathcal{M}(\mathbf{0}),l}\le C_1\|K\|_{\mathcal{M}(\mathbf{0}),l}$$

so the family $\{K_n\}$ is bounded in $\mathcal{M}(\mathbf{0})$. By Theorem 7.11, the norms of the operators $\boldsymbol{K}_n$ acting on $L^p(\mathfrak{g})$ are uniformly bounded. It remains to recall from Section 3 that, for every $f \in \mathcal{S}_0(\mathfrak{g})$,

$$\boldsymbol{K}f = \boldsymbol{K}_n f,$$

if n is large enough. Therefore, by Lemma 7.1, the convergence holds on a dense subset of $L^p(\mathfrak{g})$. □

Acknowledgements

I wish to extend thanks to Alexander Nagel, Fulvio Ricci, and Elias M. Stein for their interest in my work and an inspiring conversation. I am also indebted to François Piquard for pointing out an editorial omission in the initial version of the manuscript.

References

[1] M. Christ, The strong maximal function on a nilpotent group, *Trans. Am. Math. Soc.*, 331 (1992), 1-13,
[2] M. Duflo, Représentations de semi-groupes de mesures sur un groupe localement compact, *Ann. Inst. Fourier, Grenoble*, 28.2 (1978), 225-249,
[3] J. Duoandikoetxea and Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, *Inv. math.*, 84 (1986), 541-561,
[4] G.B. Folland and E.M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton 1982,
[5] P. Głowacki, Stable semigroups of measures as commutative approximate identities on non-graded homogeneous groups, *Inv. Math.*, 83 (1986), 557-582,
[6] P. Głowacki, The Melin calculus for general homogeneous groups, *Arkiv för matematik*, 45 (2007), 31-48,
[7] P. Głowacki, Composition and L^2-boundedness of flag kernels, *Colloq. Math.*, 118 (2010), 581-585. Correction, *Colloq. Math.*, 120 (2010), 331,
[8] P. Głowacki, Flag kernels of arbitrary order, *arXiv:1006.3440v2*,
[9] G. A. Hunt, Semigroups of measures on Lie groups, *Trans. Amer. Math. Soc.* 81 (1956), 264-293,
[10] A. Nagel, F. Ricci, and E.M. Stein, Singular integrals with flag kernels and analysis on quadratic CR manifolds, *J. Func. Analysis* 181, 29-118 (2001).
[11] A. Nagel, F. Ricci, E.M. Stein, S. Wainger, Singular integrals with flag kernels on homogeneous groups I, *arXiv: 1108.0177v1*,
[12] E.M. Stein, Harmonic analysis, *Princeton University Press*, Princeton NJ 1993,
[13] E. M. Stein, *Algebras of operators*, http://www.math.ucla.edu/dls/2009/stein.shtml,
[14] K. Yosida *Functional analysis*, Springer-Verlag, Berlin-Heidelberg-New York 1980.

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland