L^p-MULTIPLIERS ON HOMOGENEOUS GROUPS SENSITIVE TO THE GROUP STRUCTURE

PAWEŁ GŁOWACKI

Abstract. We propose new sufficient conditions for L^p-multipliers on homogeneous nilpotent groups. The multipliers generalise the flag multipliers of Nagel-Ricci-Stein-Wainger, but the approach and the techniques applied are entirely different. Our multipliers are better adapted to the specific commutation rules on the Lie algebra of the given group. The proofs are based on a new symbolic calculus fashioned after Hörmander. We also take advantage of Cotlar-Stein lemma, and Littlewood-Paley theory in the spirit of Duoandikoetxea-Rubio de Francia.

Contents

1. Introduction

Classical multiplier operators are convolution linear operators of the form

$$\mathcal{S}(\boldsymbol{R}^N) \ni f \mapsto T_K f = f \star K \in C^\infty(\boldsymbol{R}^N),$$

where

$$f \star K(x) = \int_{\boldsymbol{R}^N} f(x-y)K(y)dy = \langle K, f_x \rangle, \qquad f_x(y) = f(x-y).$$

Here K is a tempered distribution on $\boldsymbol{R}^N$ such that $m(\xi) = \widehat{K}(\xi)$ is a locally integrable function. The function m is then called a multiplier. As the space of test functions f one may adopt the Schwartz class $\mathcal{S}(\boldsymbol{R}^N)$ of smooth

2010 *Mathematics Subject Classification.* 42B15 42B20 42B25.
Key words and phrases. homogeneous groups, L^p-multipliers, Fourier transform, symbolic calculus, Hörmander metrics, singular integrals, flag kernels, Littlewood-Paley theory.

functions which decay rapidly at infinity along with all their derivatives. The primary goal of the study is the boundedness of such operators in terms of the function m. We say that a linear operator T on $\mathcal{S}(\boldsymbol{R}^N)$ is bounded in the Lebesgue norm $\|\cdot\|_{L^p}$, where $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\|Tf\|_{L^p} \le C\|f\|_{L^p}, \qquad f \in \mathcal{S}(\boldsymbol{R}^N).$$

There are numerous classical results which give (mostly) sufficient conditions on the function m that assure the boundedness of the multiplier operator. Let us mention just two such results relevant to the subject of this paper.

By the Plancherel theorem we know that the multiplier operator is bounded on $L^2(\boldsymbol{R}^N)$ if and only if the symbol m is essentially bounded. The other result is known as the Marcinkiewicz theorem and gives a sufficient condition for the boundednes on the $L^p(\boldsymbol{R}^N)$-spaces, for all $1 < p < \infty$. One of the versions of this theorem requires that m is differentiable everywhere except where $\xi_k = 0$, for some $1 \le k \le N$, and satisfies the estimates

$$|D^\alpha m(\xi)| \le C \prod_{k=1}^{N} |\xi_k|^{-\alpha_k},$$

for a constant $C > 0$ and all multiindices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, where $\alpha_k = 0$ or 1.

Similar questions have been dealt with in the context of a nilpotent homogeneous group. Instead of the usual addition of vectors $(x, y) \mapsto x + y$ in $\boldsymbol{R}^N$ we consider a group multiplication

$$(x, y) \mapsto xy = x + y + P(x, y),$$

where $P : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a polynomial mapping

$$P(x, y) = \Big(P_1(x, y), P_2(x, y), \dots, P_N(x, y)\Big)$$

with terms of order at least two. For every j, the polynomial P_j depends only on the variables x_k, y_k, where $1 \le k < j$. Furthermore,

$$P(x, 0) = P(0, x) = P(x, -x) = 0,$$

for every $x \in \boldsymbol{R}^N$. Additionally, we assume that our group is homogeneous, that is, there exist numbers $1 \le p_1 \le p_2 \le \cdots \le p_N$ such that the mappings

$$\delta_t x = \Big(t^{p_1} x_1, t^{p_2} x_2, \dots, t^{p_N} x_N\Big)$$

are group automorphisms called dilations. Such a group is necessarily connected, simply connected, and nilpotent. The simplest and best known noncommutative group of this type is the Heisenberg group, where the polynomial multiplication is defined on $\boldsymbol{R}^3$ in the following way

$$xy = \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\Big).$$

The mappings

$$\delta_t x = \Big(t^{p_1} x_1, t^{p_2} x_2, t^{p_3} x_3\Big), \qquad t > 0,$$

are automorphic dilations if and only if $p_1 + p_2 = p_3$.

The multiplier operators on $(\boldsymbol{R}^N, P)$ are convolution operators

$$\mathcal{S}(\boldsymbol{R}^N) \ni f \mapsto f \star K \in C^\infty(\boldsymbol{R}^N),$$

where

$$f \star K(x) = \int_{\boldsymbol{R}^N} f(xy^{-1})K(y)dy = \langle K, f_x \rangle, \qquad f_x(y) = f(xy^{-1}),$$

for a tempered distribution K on $\boldsymbol{R}^N$. Again, the question is: What are the conditions we may impose on the Fourier transform $\widehat{K}$ that are sufficient for the operator $f \mapsto f \star K$ to be bounded on $L^p(\boldsymbol{R}^N)$? Various examples in the theory of pseudodifferential operators which is intimately connected to the analysis on the higher-dimensional Heisenberg groups show that the conditions we know from the classical theory of multipliers have to be substantially strengthened. For instance, the boundedness of $\widehat{K}$ is no longer sufficient for the boundedness of T_K on $L^2(\boldsymbol{R}^N, P)$. The weakest sufficient conditions for the L^p-boundedness have been obtained by Nagel-Ricci-Stein-Wainger [16]. Here the smoothness of $\widehat{K}$ is assumed where $\xi_N \neq 0$ and it is required that

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k=1}^{N} \Big(\sum_{j \geq k} |\xi_j|^{1/p_j} \Big)^{-p_k \alpha_k},$$

for $|\alpha| \leq M$, where M is sufficiently large. (Actually, this is a somewhat simplified version of the condition. For a full version, see Example 8.10 below.) The operator norm of the multiplier operator depends on a finite number of constants C_α.

Let us pause here to reflect on the occurrence of the variables with higher indexes in the above estimates. This is a consequence of the noncommutativity of the group and is closely related to the fact that the polynomials P_k defining the multiplication depend on the "earlier" variables. Turning the matter around we may say that each variable x_j or y_j occurs only in polynomials P_k, for $k > j$. This suggests the idea of a new order in the set

$$\mathcal{N} = \{k \in \boldsymbol{N} : 1 \leq k \leq N\}.$$

Let us write $k \prec j$ if P_j really depends on x_k or y_k. This relation can be extended to a partial order in $\mathcal{N}$. Examples show that the order may be much coarser than the natural linear order in $\mathcal{N}$.

In this paper we wish to offer an improvement on the described above results. The condition for the L^2-boundedness is weakened to

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k \in \mathcal{N} \setminus \mathcal{N}_{\max}} \Big(\sum_{j \succ k} |\xi_j|^{1/p_j} \Big)^{-p_k \alpha_k}, \tag{1.1}$$

where $\mathcal{N}_{\max}$ consists of the maximal elements of $\mathcal{N}$ with regard to the partial order $\prec$. The condition for the L^p-boundedness takes the form

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k \in \mathcal{N}} \Big(\sum_{j \succeq k} |\xi_j|^{1/p_j} \Big)^{-p_k \alpha_k}. \tag{1.2}$$

In (1.1) as well as in (1.2) it is assumed that $\xi_j \neq 0$ if $j \in \mathcal{N}_{\max}$. As before, the norms of the multiplier operators depend on a finite number of constants C_α. The flag kernels of Nagel-Ricci-Stein-Wainger [16] satisfy (1.2).

These conditions are not only weaker but also seem much better adjusted to the individual group structure. For example, if the group is Abelian, the conditions (1.2) easily reduce to the classical Marcinkiewicz C^∞-conditions.

Our approach is based on a new symbolic calculus, where the typical estimates look like

$$|D^\alpha a(\xi)| \leq C_\alpha \prod_{k\in\mathcal{N}} g_k(\xi)^{-p_k\alpha_k}, \qquad \xi \in (\boldsymbol{R}^N)^\star. \tag{1.3}$$

In the most interesting cases the weight functions g_k do not depend on the independent variables, that is the variables ξ_j, where $k \not\preceq j$. If, for instance,

$$g_k(\xi) = 1 + \sum_{j\succeq k} |\xi_j|^{1/p_j},$$

then by differentiating with respect to the variable ξ_k no gain is produced in the independent directions.

The idea of the calculus goes back to Hörmander [12] and Melin [14] (see, Manchon [13] or Głowacki [8]). Central to the method is the formula

$$(f \star g)^\wedge(\xi) = \iint_{\boldsymbol{R}^N\times\boldsymbol{R}^N} e^{-i\langle x+y,\xi\rangle} e^{-i\langle P(x,y),\xi\rangle} f(x)g(y)dxdy,$$

for Schwartz functions f, g. This calculus seems to be well suited to the approximation of "symbols" like those in (1.2) by symbols (1.3) of the calculus and transferring, perhaps in a weaker form, some of the properties of the latter. In particular, it is shown that if distributions K_1, K_2 satisfy (1.2), then they are convolvable (see Definition 10.3) and the convolution $K_1 \star K_2$ also satifies (1.2). This is proved without using the fact that the corresponding operators are bounded on L^2 and extends to similar classes of distributions of order different from zero.

This paper is heavily influenced by Duoandikoetxea-Rubio de Francia [4], Hörmander [12], and Melin [14]. The idea of $\mathfrak{N}$-kernels has been inspired by the concept of flag kernels of Nagel-Ricci-Stein-Wainger [16].

2. Basic Setup

Let X be a real N-dimensional vector space with a fixed linear basis $\{e_k\}_{k=1}^N$. Accordingly, each element $x \in X$ has a representation as

$$x = (x_1, x_2, \dots, x_N).$$

Occasionally, it will be more convenient to write

$$x = (x_k)_{k\in\mathcal{N}}, \qquad \mathcal{N} = \{1, 2, \dots, N\}.$$

The space X is assumed to be homogeneous, that is, endowed with a family of dilations $\{\delta_t\}$. The vectors in the basis are supposed to be invariant under dilations:

$$\delta_t e_k = t^{p_k} e_k, \qquad t > 0,\ k \in \mathcal{N},$$

where $1 \leq p_1 \leq p_2 \leq \cdots \leq p_N$. The number $Q = \sum_{k=1}^N p_k$ is called *the homogeneous dimension* of X. We have

$$d\delta_t x = t^Q dx, \qquad t > 0.$$

The function

$$\rho(x) = \sum_{k=1}^{N} |x_k|^{1/p_k}, \qquad x \in X,$$

will play the role of *a homogeneous norm* on X. We also choose and fix the l^1-norm

$$\|x\| = \|\sum_{k=1}^{N} x_k e_k\| = \sum_{k=1}^{N} |x_k|. \tag{2.1}$$

By $\alpha, \beta, \dots, A, B, \dots, a, b, \cdots$ we shall denote multiindices in $\mathcal{A}_{\boldsymbol{N}} = \boldsymbol{N}^N$, where $\boldsymbol{N}$ stands for the set of all nonnegative integers. Let

$$|\alpha| = \sum_{k=1}^{N} \alpha_k. \qquad p(\alpha) = \sum_{k=1}^{N} p_k \alpha_k.$$

The set of multiinices $M = (m_1, m_2, \dots, m_N)$, where $m_k \in \boldsymbol{R}$ will be denoted by $\mathcal{A}_{\boldsymbol{R}}$. Here $\boldsymbol{R}$ denotes the field of real numbers.

We shall adopt the following notation for partial derivatives:

$$D_k = D_{x_k} = \frac{\partial}{\partial x_k}, \qquad D^\alpha = \prod_k D_k^{\alpha_k}.$$

For a function f on X we shall write

$$\widetilde{f}(x) = f(-x), \qquad x \in X.$$

The Schwartz space of smooth functions which vanish rapidly at infinity along with all their derivatives will be denoted by $\mathcal{S}(X)$. The seminorms

$$f \to \max_{|\alpha|+|\beta| \le m} \sup_{x \in X} |x^\alpha D^\beta f(x)|, \qquad m \in \boldsymbol{N},$$

form a complete set of seminorms in $\mathcal{S}(X)$ giving it a structure of a locally convex Fréchet space. Needless to say that the seminorms are actually norms. The subspace $C_c^\infty(X)$ of functions with compact support is dense in $\mathcal{S}(X)$. By $L^1(X)$ and $L^2(X)$ we denote the usual Lebesgue spaces. $\mathcal{S}(X)$ is a dense subspace of the Lebesgue spaces.

Analogous notation will be applied to the objects on the dual space $X^\star$ with the dual basis $\{e_k^\star\}_{k \in \mathcal{N}}$ and the dual dilations still denoted by $\{\delta_t\}_{t>0}$.

The Fourier transforms are denoted by $f \mapsto f^\wedge$ and $g \mapsto g^\vee$. We choose Lebesgue measures dx in X and $d\xi$ in $X^\star$ so that

$$f^\wedge(\xi) = \widehat{f}(\xi) = \int_X f(x) e^{-i\langle x, \xi \rangle} dx, \qquad g^\vee(x) = \int_{X^\star} g(\xi) e^{i\langle x, \xi \rangle} d\xi,$$

where $f \in \mathcal{S}(X),\ g \in \mathcal{S}(X^\star)$ and

$$\langle x, \xi \rangle = \sum_{k=1}^{N} x_k \xi_k$$

is the duality of vector spaces. If P is a polynomial on X, then

$$\big(Pf\big)^\wedge = P(iD)\widehat{f},$$

for $f \in \mathcal{S}(X)$.

By $|M|$ we denote the cardinality of a finite set M.

Let A, B be positive quantities. We shall write $A \lesssim B$ to say that there exists a constant $c > 0$ whose precise value is irrelevant such that $A \leq cB$.

Some more notation is explained in Appendix (Section 10) or in the current text of the paper.

3. Metrics

Let $\prec$ be a partial order in $\mathcal{N} = \{1, 2, \ldots, N\}$ such that $k \prec j$ implies $k < j$. Instead of $k \prec j$ we shall also write $j \succ k$. The expressions $k \preceq j$ and $j \succeq k$ mean that $k \prec j$ or $k = j$.

Another basic structure in $\mathcal{N}$ which is going to be intrumental throughout the paper is filtration. A family $\mathfrak{N} = \{\mathcal{N}_k\}_{k \in \mathcal{N}}$ of subsets of $\mathcal{N}$ is called *a filtration* if for every $k \in \mathcal{N}$ and every $j \in \mathcal{N}$,

a) $j \succ k$ implies $j \in \mathcal{N}_k$.

b) $j \in \mathcal{N}_k$ implies $\mathcal{N}_j \subseteq \mathcal{N}_k$.

If, for every $k \in \mathcal{N}$, $k \in \mathcal{N}_k$, we say that the filtration is *closed.*

Any filtration $\mathfrak{N} = \{\mathcal{N}_k\}_{k \in \mathcal{N}}$ determines a set of partial homogeneous norms

$$N_k(\xi) = \sum_{j \in \mathcal{N}_k} |\xi_j|^{1/p_j}, \qquad \xi \in X^\star. \tag{3.1}$$

We shall consider *metrics*, that is, families of norms on $X^\star$ of the form

$$\mathbf{g}_\xi(\eta) = \sum_{k=1}^{N} \frac{|\eta_k|}{g_k(\xi)^{p_k}}, \qquad \xi \in X^\star, \eta \in X^\star, \tag{3.2}$$

where g_k are continuous strictly positive functions. Every filtration $\mathfrak{N}$ determines a metric $\mathbf{g} = \mathbf{g}_{\mathfrak{N}}$, where

$$g_k(\xi) = 1 + N_k(\xi), \qquad \xi, \eta \in X^\star.$$

Definition 3.3. This class of metrics will be denoted by $\mathcal{G}$.

With few exceptions, these are the metrics we are going to consider here (see Remark 3.19 below). The metric corresponding to the filtration $\{\mathcal{N}_k\}$, where

$$N_k = \{j \in \mathcal{N} : j \succ k\}$$

is special. We will denote it by $\mathbf{q}$ and will refer to it as *the basic metric* on $X^\star$. We have

$$\mathbf{q}_\xi(\eta) = \sum_{k=1}^{N} \frac{|\eta_k|}{q_k(\xi)^{p_k}}, \tag{3.4}$$

where

$$q_k(\xi) = 1 + \sum_{j \succ k} |\xi_j|^{1/p_j}, \qquad k \in \mathcal{N},\ \xi \in X^\star.$$

The following proposition says that the metric $\mathbf{q}$ is *self-tempered.*

Proposition 3.5. *There exist $C_0, T > 0$ such that, for all $\xi, \eta, \zeta \in X^\star$,*

$$\left(\frac{\mathbf{q}_\xi(\zeta)}{\mathbf{q}_\eta(\zeta)}\right)^{\pm 1} \leq C_0\big(1 + \mathbf{q}_\xi(\xi - \eta)\big)^T.$$

Proof. It is sufficient to show that, for every $k \in \mathcal{N}$, there exists $T_k > 0$ such that

$$\left(\frac{q_k(\xi)}{q_k(\eta)}\right)^{\pm 1} \lesssim \left(1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|}{q_j(\xi)^{p_j}}\right)^{T_k}, \tag{3.6}$$

for $\xi, \eta, \zeta \in X^\star$. Let us prove (3.6). We have

$$\begin{aligned}\frac{q_k(\xi)}{q_k(\eta)} &\lesssim 1 + q_k(\eta)^{-1} \sum_{j \succ k} |\xi_j - \eta_j|^{1/p_j} \\ &\lesssim 1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|^{1/p_j}}{q_j(\eta)} \lesssim 1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|}{q_j(\eta)^{p_j}}.\end{aligned}$$

The other part is proved by reverse induction. If k is a maximal element with respect to $\prec$, then

$$q_k(\xi) = q_k(\eta) = 1.$$

If not, let us assume that

$$\frac{q_j(\xi)}{q_j(\eta)} \lesssim \left(1 + \sum_{l \succ j} \frac{|\xi_l - \eta_l|}{q_l(\xi)^{p_l}},\right)^{T_j} \tag{3.7}$$

for $j \succ k$. As in the first part of the proof,

$$\frac{q_k(\xi)}{q_k(\eta)} \lesssim 1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|}{q_j(\xi)^{p_j}} \cdot \left(\frac{q_j(\xi)}{q_k(\eta)}\right)^{p_j},$$

so that, by (3.7) and $q_j(\eta) \leq q_k(\eta)$,

$$\begin{aligned}\frac{q_k(\xi)}{q_k(\eta)} &\lesssim \left(1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|}{q_j(\xi)^{p_j}}\right)\left(1 + \sum_{l \succ j} \frac{|\xi_l - \eta_l|}{q_l(\xi)^{p_l}}\right)^{p_N T_j} \\ &\lesssim \left(1 + \sum_{j \succ k} \frac{|\xi_j - \eta_j|}{q_j(\xi)^{p_j}}\right)^{T_k},\end{aligned} \tag{3.8}$$

where $T_k = p_N \max_{j \succ k} T_j + 1$, which implies (3.6). ☐

Corollary 3.9. *The metric* $\mathbf{q}$ *is* slowly varying*, that is, if* $\mathbf{q}_\xi(\xi - \eta) < 1$*, then*

$$\left(\frac{\mathbf{q}_\xi(\zeta)}{\mathbf{q}_\eta(\zeta)}\right)^{\pm 1} \leq C_1, \tag{3.10}$$

for some $C_1 \geq 1$.

Corollary 3.11. *Let* $\widetilde{q}_k(\xi) = |\xi_k|^{1/p_k} + q_k(\xi)$*,* $k \in \mathcal{N}$*,* $\xi \in X^\star$*. For every* $k \in \mathcal{N}$*, there exists* $R_k > 0$ *such that*

$$\left(\frac{\widetilde{q}_k(\xi)}{\widetilde{q}_k(\eta)}\right)^{\pm 1} \lesssim \left(1 + \sum_{j \succeq k} \frac{|\xi_j - \eta_j|}{q_j(\xi)^{p_j}}\right)^{R_k},$$

The metrics $\mathbf{g} \in \mathcal{G}$ are $\mathbf{q}$-tempered:

Proposition 3.12. *Let $\mathbf{g} \in \mathcal{G}$. Then, for every $k \in \mathcal{N}$, $q_k \le g_k$, and, for all $\xi, \eta, \zeta \in X^\star$,*

$$\left(\frac{\mathbf{g}_\xi(\zeta)}{\mathbf{g}_\eta(\zeta)}\right)^{\pm 1} \lesssim \big(1 + \mathbf{q}_\xi(\xi - \eta)\big)^T,$$

where $T > 0$ is as in Proposition 3.5.

Proof. We note first that $q_k \le g_k$, since the sets N_k for the basic metric $\mathbf{q}$ are, by definition, minimal. Now, observe that, for every $k \in \mathcal{N}$,

$$\frac{g_k(\xi)}{g_k(\eta)} \le 1 + \frac{g_k(\eta - \xi)}{g_k(\eta)}.$$

If $j \in \mathcal{N}_k$, then by the filtration property, $\mathcal{N}_j \subseteq \mathcal{N}_k$, hence $q_j \le g_j \le g_k$. Therefore,

$$\begin{aligned} 1 + \frac{g_k(\eta - \xi)}{g_k(\eta)} &\le 1 + \sum_{j \in \mathcal{N}_k} \frac{|\eta_j - \xi_j|^{1/p_j}}{q_j(\eta)} \\ &\lesssim 1 + \sum_{j \in \mathcal{N}} \frac{|\eta_j - \xi_j|}{q_j(\eta)^{p_j}} = 1 + \mathbf{q}_\eta(\eta - \xi), \end{aligned}$$

which implies

$$\frac{\mathbf{g}_\xi(\zeta)}{\mathbf{g}_\eta(\zeta)} \lesssim 1 + \mathbf{q}_\xi(\xi - \eta).$$

By the above and Proposition 3.5,

$$\frac{\mathbf{g}_\eta(\zeta)}{\mathbf{g}_\xi(\zeta)} \lesssim 1 + \mathbf{q}_\eta(\eta - \xi) \lesssim \big(1 + \mathbf{q}_\xi(\xi - \eta)\big)^T,$$

which completes our proof. □

Let $\mathbf{g}$ be a metric (not necessarily in $\mathcal{G}$) and let $\boldsymbol{m}$ be a strictly positive function on $X^\star$. For a smooth function f on $X^\star$, $\xi \in X^\star$ and $s \ge 0$, let

$$|f|_s^{\boldsymbol{m},\mathbf{g}}(\xi) = \boldsymbol{m}(\xi)^{-1} \max_{p(\alpha) \le s} \mathbf{g}^\alpha(\xi)|D^\alpha f(\xi)|, \tag{3.13}$$

where

$$\mathbf{g}^\alpha(\xi) = \prod_{k=1}^N g_k(\xi)^{p_k \alpha_k}. \tag{3.14}$$

Let also

$$|f|_s^{\boldsymbol{m},\mathbf{g}} = \sup_{\xi \in X^\star} |f|_s^{\boldsymbol{m},\mathbf{g}}(\xi).$$

Instead of $|\cdot|_s^{\mathbf{1},\mathbf{g}}$ we shall write $|\cdot|_s^{\mathbf{g}}$. Let

$$S(\boldsymbol{m}, \mathbf{g}) = \{f \in C^\infty(X^\star) : \forall_{s \ge 0}\ |f|_s^{\boldsymbol{m},\mathbf{g}} < \infty\}.$$

The space $S(\boldsymbol{m}, \mathbf{g})$ with the seminorms $|\cdot|_s^{\boldsymbol{m},\mathbf{g}}$ is a Fréchet space.

Let $\mathbf{g}_1, \mathbf{g}_2$ be metrics and $\boldsymbol{m}_1, \boldsymbol{m}_2$ strictly positive functions. A linear mapping

$$T : S(\boldsymbol{m}_1, \mathbf{g}_1) \to S(\boldsymbol{m}_2, \mathbf{g}_2) \tag{3.15}$$

is Fréchet continuous, if for every n, there exist k such that

$$|Tf|_n^{\boldsymbol{m}_2,\mathbf{g}_2} \lesssim |f|_k^{\boldsymbol{m}_1,\mathbf{g}_1}, \qquad f \in S(\boldsymbol{m}_1, \mathbf{g}_1).$$

The product $S(\boldsymbol{m}_1, \mathbf{g}) \times S(\boldsymbol{m}_2, \mathbf{g})$ is a Fréchet space with the product topology. We shall also consider Fréchet continuous bilinear mappings

$$T : S(\boldsymbol{m}_1, \mathbf{g}) \times S(\boldsymbol{m}_2, \mathbf{g}) \to S(\boldsymbol{m}_3, \mathbf{g}). \tag{3.16}$$

We need, however, still another concept of convergence. A sequence $f_n \in S(\boldsymbol{m}, \mathbf{g})$ is said to be *weakly convergent* if it is bounded and pointwise convergent (see Hörmander [12], page 369). Note that, by the Ascoli theorem, for a bounded sequence, pointwise convergence is equivalent to the convergence in C^∞-topology: for each α, the sequence $D^\alpha f_n$ is uniformly convergent on compact subsets of $X^\star$.

A linear mapping (3.15) is said to be *weakly continuous* if the weak convergence $f_n \to f$ in $S(\boldsymbol{m}_1, \mathbf{g}_1)$ implies the weak convergence $Tf_n \to Tf$ in $S(\boldsymbol{m}_2, \mathbf{g}_2)$. A bilinear mapping (3.16) is said to be *weakly continuous* if the weak convergence $f_n \to f$ in $S(\boldsymbol{m}_1, \mathbf{g})$ together with the weak convergence $g_n \to g$ in $S(\boldsymbol{m}_2, \mathbf{g})$ imply the weak convergence $T(f_n, g_n) \to T(f, g)$ in $S(\boldsymbol{m}_3, \mathbf{g}_2)$.

Proposition 3.17. *Let $\mathbf{g}$ be a metric and $\boldsymbol{m}$ a strictly positive function on $X^\star$. The subspace $C_c^\infty(X^\star)$ is weakly dense in $S(\boldsymbol{m}, \mathbf{g})$.*

Proof. Let $\psi \in C_c^\infty(X^\star)$ be equal to 1 in a neighbourhood of the origin. Let $\psi_n(\xi) = \psi(\delta_{1/n}\xi)$. Let $f \in S(\boldsymbol{m}, \mathbf{g})$. Of course, the sequence $\psi_n f$ is pointwise convergent to f, so we only have to show that it is bounded in $S(\boldsymbol{m}, \mathbf{g})$. In fact, if $p(\alpha) \le s$, then

$$\begin{aligned} &\boldsymbol{m}(\xi)^{-1}\mathbf{g}^\alpha(\xi)|D^\alpha(\psi_n f)(\xi)| \\ &\quad \lesssim \sum_{\beta+\gamma=\alpha} (1+\rho(\xi))^{p(\beta)} n^{-p(\beta)} \boldsymbol{m}(\xi)^{-1}\mathbf{g}^\gamma(\xi)|D^\gamma f(\xi)| \\ &\quad \le |f|_s^{\boldsymbol{m},\mathbf{g}} \end{aligned}$$

since $\rho(\xi) \approx n$ on the support of $D^\beta\psi_n$, if $\beta \ne 0$. If $\beta = 0$, the estimate is obvious. ☐

We denote by $\mathcal{M}(\mathbf{q})$, the class of strictly positive functions $\boldsymbol{m}$ on $X^\star$ such that there exists $T_1 > 0$ such that, for all $\xi, \eta \in X^\star$,

$$\left(\frac{\boldsymbol{m}(\xi)}{\boldsymbol{m}(\eta)}\right)^{\pm 1} \lesssim \big(1 + \mathbf{q}_\xi(\xi - \eta)\big)^{T_1}.$$

We may express this property, by saying that the functions $\boldsymbol{m}$ are $\mathbf{q}$-*tempered.*

The elements of $\mathcal{M}(\mathbf{q})$ will be called $\mathbf{q}$-*weights* or simply *weights.* Observe also that if $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}(\mathbf{q})$, then $\boldsymbol{m}_1\boldsymbol{m}_2 \in \mathcal{M}(\mathbf{q})$ and $\boldsymbol{m}_1^\theta \in \mathcal{M}(\mathbf{q})$, for every $\theta \in \boldsymbol{R}$.

Proposition 3.18. *Let $\mathbf{g} \in \mathcal{G}$. Let $M = (m_1, m_2, \ldots, m_N) \in \mathcal{A}_{\boldsymbol{R}}$ and let*

$$\boldsymbol{m}_M(\xi) = \prod_{k=1}^N g_k(\xi)^{m_k},$$

Then, $\boldsymbol{m}_M$ is a $\mathbf{q}$-weight. In particular, for every $\alpha \in \mathcal{A}_{\boldsymbol{N}}$, $\mathbf{g}^\alpha \in \mathcal{M}(\mathbf{q})$.

Proof. It follows from Proposition 3.12 that functions g_k are $\mathbf{g}$-weights. Thus, by the preceding remark $\boldsymbol{m}_M \in \mathcal{M}(\mathbf{g})$, for every $M \in \mathcal{A}_{\boldsymbol{R}}$. ☐

Let $\mathbf{g} \in \mathcal{G}$. The metric $\mathbf{g} \oplus \mathbf{g}$ on $X^\star \times X^\star$ is defined by

$$(\mathbf{g} \oplus \mathbf{g})_{(\xi_1,\xi_2)}(\eta_1, \eta_2) = \mathbf{g}_{\xi_1}(\eta_1) + \mathbf{g}_{\xi_2}(\eta_2), \qquad \xi_1, \xi_2, \eta_1, \eta_2 \in X^\star.$$

This metric no longer belongs to $\mathcal{G}$. For $f \in \mathcal{S}(X^\star \times X^\star)$ and a strictly positive function $\boldsymbol{m}$ on $X^\star \times X^\star$, let

$$|f|_s^{\boldsymbol{m},\mathbf{g}\oplus\mathbf{g}} = \sup_{\xi,\eta} \sup_{p(\alpha)+p(\beta)\le s} \boldsymbol{m}(\xi,\eta)^{-1} \mathbf{g}^\alpha(\xi)\mathbf{g}^\beta(\eta)|D_\xi^\alpha D_\eta^\beta f(\xi,\eta)|.$$

Remark 3.19. Througout the paper, the metric $\mathbf{g} \oplus \mathbf{g}$, where $\mathbf{g} \in \mathcal{G}$, is the only instance of a metric which is not in $\mathcal{G}$. In addition, it will only play an auxiliary role.

4. A partition of unity

Let v be a norm on $X^\star$. Denote

$$B_v(a, r) = \{\xi \in X^\star : v(\xi - a) < r\},$$

for $a \in X^\star$, $r > 0$. The following is a simplified lemma of Hörmander ([12], Lemma 2.5). We adapt the original proof to our needs.

Proposition 4.1. *There exists a discrete set $\mathcal{B} = \{b_\nu : \nu \in \boldsymbol{N}\} \subseteq X^\star$ such that the family of the balls $A_\nu = B_{\mathbf{q}_{b_\nu}}(b_\nu, 1/2)$ is a covering of $X^\star$, and no point $\xi \in X^\star$ can belong to more than $(4C_1^3+1)^N$ larger balls $B_\nu = B_{\mathbf{q}_{b_\nu}}(b_\nu, 1)$. There also exists a sequence of functions $\varphi_\nu \in C_c^\infty(B_\nu)$ bounded in $S(\mathbf{1}, \mathbf{q})$ and such that for every $\xi \in X^\star$*

$$\sum_\nu \varphi_\nu(\xi) = 1.$$

Furthermore, there exist constants $m, M > 0$ such that

$$\sum_{\nu\in\boldsymbol{N}} \Big(1 + \mathbf{q}_{b_\nu}(\xi - b_\nu)\Big)^{-m} \le M, \tag{4.2}$$

for every $\xi \in X^\star$.

Proof. Let C_1 be as in (3.10). Let $\{b_\nu\}$ be a maximal sequence of points in $X^\star$ such that

$$\mathbf{q}_{b_\nu}(b_\mu - b_\nu) \ge \frac{1}{2C_1}, \qquad \mu \ne \nu. \tag{4.3}$$

Let $\xi \in X^\star$. Note that

$$\mathbf{q}_\xi(\xi - b_\nu) < 1/2C_1 \quad \text{implies} \quad \mathbf{q}_{b_\nu}(\xi - b_\nu) < 1/2.$$

Therefore, either $\mathbf{q}_{b_\nu}(\xi - b_\nu) < 1/2$ for some ν, or

$$\mathbf{q}_\xi(\xi - b_\nu) \ge \frac{1}{2C_1} \quad \text{and} \quad \mathbf{q}_{b_\nu}(\xi - b_\nu) \ge 1/2 \ge \frac{1}{2C_1}.$$

The latter contradicts the maximality of our sequence. The former implies that $\{A_\nu\}_{\nu\in\boldsymbol{N}}$ is a covering, which proves the first statement of the proposition.

To show that the covering $\{B_\nu\}_{\nu\in\boldsymbol{N}}$ is uniformly locally finite take a $\xi \in X^\star$ and let

$$M(\xi) = \{\nu : \xi \in B_\nu\}.$$

If $\nu \in M(\xi)$, then $\mathbf{q}_{b_\nu}(\xi - b_\nu) < 1$, which implies $\mathbf{q}_\xi(\xi - b_\nu) < C_1$. On the other hand

$$\mathbf{q}_\xi(b_\mu - b_\nu) \geq \mathbf{q}_{b_\nu}(b_\nu - b_\mu)/C_1 \geq 1/2C_1^2,$$

for $\mu \neq \nu$. Thus, for every $\nu \in M(\xi)$,

$$B_{\mathbf{q}_\xi}(b_\nu, 1/4C_1^2) \subseteq B_{\mathbf{q}_\xi}(\xi, C_1 + 1/4C_1^2),$$

and the balls $B_{\mathbf{q}_\xi}(b_\nu, 1/4C_1^2\}$ are pairwise disjoint, so

$$|M(\xi)| \leq \left(\frac{C_1 + 1/4C_1^2}{1/4C_1^2}\right)^N = (4C_1^3 + 1)^N,$$

as claimed.

Let $\psi \in C_c^\infty\Big(B_{\|\cdot\|}(0,1)\Big)$ be equal to 1 on the smaller ball $B_{\|\cdot\|}(0,1/2)$. Let

$$\psi_\nu(\xi) = \psi\left(\Delta_\nu^{-1}(\xi - b_\nu)\right),$$

where

$$\Delta_\nu \xi = \Big(q_k(b_\nu)^{p_k} \xi_k\Big)_{k \in \mathcal{N}}. \tag{4.4}$$

By (3.10),

$$\begin{aligned}\mathbf{q}^\alpha(\xi)|D^\alpha \psi_\nu(\xi)| &= \mathbf{q}^\alpha(\xi)\mathbf{q}^{-\alpha}(b_\nu)|D^\alpha \psi(\Delta_\nu^{-1}(\xi - b_\nu)| \\ &\leq C_1^{p(\alpha)} \sup_{\|\eta\| \leq 1} |D^\alpha \psi(\eta)| = C_\alpha,\end{aligned}$$

for all ν and all α, which shows that $\psi_\nu \in S(\mathbf{1}, \mathbf{q})$ with uniform bounds.

Since $\{A_\nu\}$ is a covering, $\sum_\mu \psi_\mu(\xi) \geq 1$ for every $\xi \in X^\star$, and it is not hard to see that the sequence

$$\varphi_\nu(\xi) = \frac{\psi_\nu(\xi)}{\sum_\mu \psi_\mu(\xi)}$$

is a partition of unity.

It remains to prove (4.2). Given $\xi \in X^\star$ and $k \in \boldsymbol{N}$, let

$$M_k(\xi) = \{\nu : \mathbf{q}_{b_\nu}(\xi - b_\nu) < k\}.$$

For every $\nu \in M_k(\xi)$,

$$\mathbf{q}_\xi(\xi - b_\nu) \leq C_0 k(1+k)^T \leq C_0(1+k)^{T+1},$$

wher C_0 is as in Proposition 3.12. Furthermore, for $\nu, \mu \in M_k(\xi)$ such that $b_\nu \neq b_\mu$,

$$\mathbf{q}_\xi(b_\nu - b_\mu) > C_0^{-1}\mathbf{q}_{b_\nu}(b_\nu - b_\mu)(1+k)^{-T} > \frac{1}{2C_0C_1(1+k)^T},$$

so the balls

$$B_{\mathbf{q}_\xi}\left(b_\nu, \frac{1}{4C_0C_1(1+k)^T}\right)$$

are mutually disjoint and contained in the ball $B_{\mathbf{q}_\xi}(\xi, 2C_0(1+k)^{T+1})$. Therefore,

$$|M_k(\xi)| \leq (8C_0^2C_1(1+k)^{2T+1})^N \lesssim (1+k)^{(2T+1)N}.$$

Finally,

$$\sum_{\nu\in\boldsymbol{N}}\Big(1+\mathbf{q}_{b_\nu}(\xi-b_\nu)\Big)^{-m}\le|M_1(\xi)|+\sum_{k=1}^{\infty}(1+k)^{-m}\Big|M_{k+1}(\xi)\setminus M_k(\xi)\Big|$$
$$\lesssim 1+\sum_{k=1}^{\infty}(1+k)^{-m+(2T+1)N}\le M,$$

if $m>(2T+1)N+1$. □

Let

$$d_\nu(\xi)=1+\mathbf{q}_{b_\nu}(b_\nu-\xi),\qquad \nu\in\boldsymbol{N},\quad \xi\in X^\star.$$

Proposition 4.5. *We have*

$$\Big(1+\mathbf{q}_\xi(\xi-b_\nu)\Big)^{\frac{1}{T+1}}\lesssim d_\nu(\xi)\lesssim\Big(1+\mathbf{q}_\xi(\xi-b_\nu)\Big)^{T+1}$$

uniformly in ξ and ν.

Proof. In fact,

$$1+\mathbf{q}_\xi(\xi-b_\nu)\lesssim 1+\mathbf{q}_{b_\nu}(b_\nu-\xi)\Big(1+\mathbf{q}_{b_\nu}(b_\nu-\xi)\Big)^T\le d_\nu(\xi)^{T+1},$$

for $\xi\in X^\star$. The other inequality is proved in a similar way. □

Let

$$d_{\nu\mu}=\max\Big\{d_\nu(b_\mu),d_\mu(b_\nu)\Big\}.$$

Corollary 4.6. *We have*

$$d_{\nu\mu}\lesssim d_\nu(\xi)^{2T+1}d_\mu(\xi)^{2T+1}$$

uniformly for ν,μ and ξ.

Proof. In fact,

$$1+\mathbf{q}_\xi(b_\nu-b_\mu)\le\Big(1+\mathbf{q}_\xi(\xi-b_\nu)\Big)\Big(1+\mathbf{q}_\xi(\xi-b_\mu)\Big), \tag{4.7}$$

so, by (4.7) and Proposition 4.5,

$$d_\nu(b_\mu)=1+\mathbf{q}_{b_\nu}(b_\nu-b_\mu)\lesssim\Big(1+\mathbf{q}_\xi(b_\nu-b_\mu)\Big)\Big(1+\mathbf{q}_{b_\nu}(b_\nu-\xi)\Big)^T$$
$$\lesssim\Big(1+\mathbf{q}_\xi(\xi-b_\nu)\Big)\Big(1+\mathbf{q}_\xi(\xi-b_\mu)\Big)d_\nu(\xi)^T$$
$$\lesssim d_\nu(\xi)^{T+1}d_\mu(\xi)^{T+1}d_\nu(\xi)^T\lesssim d_\nu(\xi)^{2T+1}d_\mu(\xi)^{2T+1}.$$

Our claim follows by symmetry. □

5. The Melin operator

We specify the abstract structure of X. From now on $X=\mathfrak{g}$ is a homogeneous Lie algebra with the Campbell-Hausdorff multiplication (see Corwin-Greenleaf [3])

$$xy=x+y+P(x,y), \tag{5.1}$$

where P is a polynomial mapping with non-zero terms of order at least 2.

We assume there is a fixed family of automorphic dilations on $\mathfrak{g}$. By definition, *dilations* of a Lie algebra are automorphisms of the form

$$\delta_t = t^{\mathcal{P}}, \qquad t > 0,$$

where $\mathcal{P} : \mathfrak{g} \to \mathfrak{g}$ is a diagonalizable linear operator with positive eigenvalues

$$0 < p_1 \leq p_2 \leq \cdots \leq p_N,$$

listed along with their multiplicities. We assume that $p_1 \geq 1$. If $\mathcal{D} = \{p_{k_j}\}$ is a complete set of pairwise different eigenvalues, then

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}_{(k_j)},$$

where

$$[\mathfrak{g}_{(k_i)}, \mathfrak{g}_{(k_j)}] \subseteq \begin{cases} \mathfrak{g}_{(k_s)}, & if\ p_{k_i} + p_{k_j} = p_{k_s}, \\ \{0\}, & if\ p_{k_i} + p_{k_j} \notin \mathcal{D}. \end{cases}$$

Let $\{e_k\}_{k=1}^N$ be a basis of eigenvectors for $\mathcal{P}$. Of course, there is some freedom in the choice of such a basis. However, once we have chosen and fixed one, we may apply the notation of Section 2.

The mapping P commutes with the dilations, that is,

$$P(\delta_t x, \delta_t y) = \delta_t P(x, y), \tag{5.2}$$

so the dilations are also automorhisms of the group. We also have

$$D_{x_k} P_j(x, y) = D_{y_k} P_j(x, y) = 0, \qquad j \leq k, \tag{5.3}$$

where $P_j(x, y) = \langle P(x, y), e_j^\star \rangle$. From

$$\begin{aligned} -x - y - P(x, y) &= -xy = (xy)^{-1} = y^{-1}x^{-1} \\ &= -y - x + P(-y, -x), \end{aligned}$$

one gets $P(x, y) = -P(-y, -x)$, which in turn implies

$$D_{x_k} P_j \neq 0 \qquad \text{iff} \qquad D_{y_k} P_j \neq 0. \tag{5.4}$$

Definition 5.5. We define $\prec$ as the smallest order in $\mathcal{N}$ such that $D_{x_k} P_j \neq 0$ or, equivalently, $D_{y_k} P_j \neq 0$ implies $k \prec j$ (cf. (5.4)).

Remark 5.6. The partial order $\prec$ is determined by our choice of basis and is by no means canonical.

Lemma 5.7. *An integer $k \in \mathcal{N}$ is maximal if and only if the basis vector e_k is central in $\mathfrak{g}$.*

Proof. In fact, suppose that k is maximal and let $x \in \mathfrak{g}$. Since $D_{x_k} P_j(x, y) = D_{y_k} P_j(x, y) = 0$, for $j > k$, it follows that $P_j(x, e_k) = P_j(e_k, x) = 0$, for every $j \in \mathcal{N}$. Therefore, $xe_k = x + e_k = e_k x$. The converse implication is trivial. □

Lemma 5.8. *Let $\mathfrak{N} = \{\mathcal{N}_k\}_{k \in \mathcal{N}}$ be a filtration in $\mathcal{N}$. Let $\mathfrak{g}_k$ be the linear subspace of $\mathfrak{g}$ generated by the vectors e_j, $j \in \mathcal{N}_k$. Then, $\mathfrak{g}_k$ is an ideal in $\mathfrak{g}$.*

Proof. Let $j \in \mathcal{N}_k$. By the Campbell-Hausdorff formula, for any $l, s \in \mathcal{N}$,

$$\lambda_s = \langle [e_l, e_j], e_s^\star \rangle = 2D_{x_l} D_{y_j} P_s(0,0).$$

If $\lambda_s \neq 0$, then $s \succ j$, hence $s \in \mathcal{N}_k$. Therefore,

$$[e_l, e_j] = \sum_{s \in \mathcal{N}_k} \lambda_s e_s \in \mathfrak{g}_k.$$

□

Example 5.9. Let $\mathfrak{g}$ be the 6-dimensional Lie algebra with a basis $\{e_k\}_{k=1}^6$ and the nonzero commutators

$$[e_1, e_2] = e_4, \; [e_1, e_5] = e_6, \; [e_2, e_3] = e_5, \; [e_3, e_4] = -e_6.$$

As automorphic dilations one can take

$$\delta_t(x) = (tx_1, tx_2, tx_3, t^2 x_4, t^2 x_5, t^3 x_6).$$

If

$$X = \sum_{j=1}^6 x_j e_j, \qquad Y = \sum_{k=1}^6 y_k e_k,$$

then

$$\begin{aligned}[X, Y] &= (x_1 y_2 - x_2 y_1) e_4 + (x_2 y_3 - x_3 y_2) e_5 \\ &\quad + (x_1 y_5 - x_5 y_1 - x_3 y_4 + x_4 y_3) e_6,\end{aligned}$$

and

$$[X, [X, Y]] = (x_1 x_2 y_3 - x_1 x_3 y_2 - x_3 x_1 y_2 + x_3 x_2 y_1) e_6,$$

so, by the Campbell-Hausdorff formula,

$$\begin{aligned}P(x, y) &= \frac{1}{2}[X, Y] + \frac{1}{12}\Big([X, [X, Y]] + [Y, [Y, X]]\Big) \\ &= \sum_{j=1}^6 P_j(x, y) e_j,\end{aligned}$$

where

$$\begin{aligned}P_1(x, y) &= P_2(x, y) = P_3(x, y) = 0, \\ P_4(x, y) &= \frac{1}{2}(x_1 y_2 - x_2 y_1), \\ P_5(x, y) &= \frac{1}{2}(x_2 y_3 - x_3 y_2), \\ P_6(x, y) &= \frac{1}{2}(x_1 y_5 - x_5 y_1 - x_3 y_4 + x_4 y_3) \\ &\quad + \frac{1}{12}(x_1 x_2 y_3 - x_1 x_3 y_2 - x_3 x_1 y_2 + x_3 x_2 y_1 \\ &\quad + y_1 y_2 x_3 - y_1 y_3 x_2 - y_3 y_1 x_2 + y_3 y_2 x_1).\end{aligned}$$

Then,

$$q_1(\xi) = 1 + |\xi_4|^{1/2} + |\xi_6|^{1/3}, \qquad q_2(\xi) = 1 + |\xi_4|^{1/2} + |\xi_5|^{1/2} + |\xi_6|^{1/3},$$

$$q_3(\xi) = 1 + |\xi_5|^{1/2} + |\xi_6|^{1/3}, \quad q_4(\xi) = q_5(\xi) = 1 + |\xi_6|^{1/3}, \quad q_6(\xi) = 1.$$

□

Definition 5.10. We define the Melin operator on $\mathfrak{g}^\star \times \mathfrak{g}^\star$ by

$$\mathbf{U}f(\xi) = \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle x+y,\xi\rangle} e^{-i\langle P(x,y),\xi\rangle} f^\vee(x,y)\,dxdy, \tag{5.11}$$

for $f \in \mathcal{S}(\mathfrak{g}^\star \times \mathfrak{g}^\star)$.

By a linear differential operator on $\mathfrak{g}^\star$ with polynomial coefficients we understand a differential operator of the form

$$\mathcal{L}f(\xi) = \sum_\alpha p_\alpha(\xi) D^\alpha f(\xi), \qquad f \in C^\infty(\mathfrak{g}^\star),$$

where p_α are polynomials and the sum is finite. If f is a differentiable function on $\mathfrak{g}^\star \times \mathfrak{g}^\star$ we write $D_\xi f = D_1 f$ and $D_\eta f = D_2 f$ for the partial derivatives with respect to the variable $(\xi,\eta) \in \mathfrak{g}^\star \times \mathfrak{g}^\star$.

Lemma 5.12. *For every linear differential operator $\mathcal{L}$ with polynomial coefficients on $\mathfrak{g}^\star$, there exists a finite number of operators $\mathcal{L}_k$ of the same type on $\mathfrak{g}^\star \times \mathfrak{g}^\star$ such that*

$$|\mathcal{L}\mathbf{U}f(\xi)| \lesssim \sum_k \|\mathcal{L}_k f\|_{A(\mathfrak{g}^\star\times\mathfrak{g}^\star)}, \qquad f \in \mathcal{S}(\mathfrak{g}^\star \times \mathfrak{g}^\star), \quad \xi \in \mathfrak{g}^\star,$$

where

$$\|f\|_{A(\mathfrak{g}^\star\times\mathfrak{g}^\star)} = \iint_{\mathfrak{g}\times\mathfrak{g}} |f^\vee(x,y)|\,dxdy.$$

Consequently, the mapping $\mathbf{U} : \mathcal{S}(\mathfrak{g}^\star \times \mathfrak{g}^\star) \to \mathcal{S}(\mathfrak{g}^\star)$ is continuous.

Proof. Let

$$P^1_{j,k}(x,y) = D_{x_k} P_j(x,y), \qquad P^2_{j,k}(x,y) = D_{y_k} P_j(x,y)$$

and

$$\mathcal{D}^1_{j,k} = P^1_{j,k}(iD_\xi, iD_\eta) \qquad \mathcal{D}^2_{j,k} = P^2_{j,k}(iD_\xi, iD_\eta).$$

Directly from (5.11) one obtains

$$\begin{aligned} \xi_k \mathbf{U}(f)(\xi) &= \mathbf{U}(T^1_k f)(\xi) - \sum_{j\succ k} \xi_j \mathbf{U}\Big(\mathcal{D}^1_{j,k} f\Big)(\xi) \\ &= \mathbf{U}(T^2_k f)(\xi) - \sum_{j\succ k} \xi_j \mathbf{U}\Big(\mathcal{D}^2_{j,k} f\Big)(\xi), \end{aligned} \tag{5.13}$$

where

$$T^1_k f(\xi,\eta) = \xi_k f(\xi,\eta), \qquad T^2_k f(\xi,\eta) = \eta_k f(\xi,\eta).$$

We also have

$$D_k \mathbf{U} f(\xi) = \mathbf{U}(\mathcal{D}_k f)(\xi), \tag{5.14}$$

where

$$\mathcal{D}_k = D_{\xi_k} + D_{\eta_k} - iP_k(iD_\xi, iD_\eta).$$

By iteration of (5.13) and (5.14), for every linear differential operator $\mathcal{L}$ with polynomial coefficients, there exist operators $\mathcal{L}_k$ of the same type such that

$$|\mathcal{L}\mathbf{U}(f)(\xi)| \le \sum_k |\mathbf{U}(\mathcal{L}_k f)(\xi)|.$$

By (5.11),

$$|\mathbf{U}(f)(\xi)| \le \iint_{\mathfrak{g}\times\mathfrak{g}} |f^{\vee}(x,y)|dxdy = \|f\|_{A(\mathfrak{g}^\star\times\mathfrak{g}^\star)},$$

for $f \in \mathcal{S}(\mathfrak{g}^\star \times \mathfrak{g}^\star)$ and $\xi \in \mathfrak{g}^\star$. Therefore,

$$|\mathcal{L}\mathbf{U}(f)(\xi)| \le \sum_k \|\mathcal{L}_k f\|_{A(\mathfrak{g}^\star\times\mathfrak{g}^\star)}.$$

□

Lemma 5.15. *Let* $\mathbf{g} \in \mathcal{G}$. *For every* $\alpha \in \mathcal{A}_{\boldsymbol{N}}$,

$$\mathcal{D}^\alpha = \sum_{\substack{p(A)+p(B)=p(\alpha)\\ \mathbf{g}^\alpha \le \mathbf{g}^A\mathbf{g}^B}} c_{AB} D_\xi^A D_\eta^B,$$

where

$$\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1}\mathcal{D}_2^{\alpha_2}\dots\mathcal{D}_N^{\alpha_N},$$

and the symbols $\mathbf{g}^\alpha$ *have been defined in* (3.13).

Proof. It is not hard to see that if the assertion holds for α and β, then it holds for $\alpha+\beta$ as well. Therefore, it is sufficient to prove it for single derivatives $\mathcal{D}_k$. We have

$$\mathcal{D}_k = \sum_{p(A)+p(B)=p_k} c_{AB} D_\xi^A D_\beta^B. \tag{5.16}$$

If $c_{AB} \ne 0$, then $A_j = B_j = 0$ unless $j \preceq k$. The sequence g_j is decreasing, so

$$g_k^{p_k} = g_k^{p(A)} g_k^{p(B)} \le \mathbf{g}^A\mathbf{g}^B.$$

□

6. Estimates for the basic metric

In this section we only consider the basic metric $\mathbf{q}$.

Lemma 6.1. *Let* $\{B_\nu\}_{\nu\in\boldsymbol{N}}$ *be the covering of Proposition 4.1. Then,*

$$|\mathbf{U}(f)(\xi)| \lesssim |f|_{N+1}^{\mathbf{q}\oplus\mathbf{q}}, \qquad f \in C_c^\infty(B_\nu \times B_\mu),$$

uniformly in ν, μ.

Proof. We have

$$|\mathbf{U}(f)(\xi)| \lesssim \iint_{\mathfrak{g}\times\mathfrak{g}} |f^\vee(x,y)|dxdy = \iint_{\mathfrak{g}\times\mathfrak{g}} |F^\vee(x,y)|dxdy,$$

where

$$F(\xi,\eta) = f(b_\nu + \Delta_\nu\xi, b_\mu + \Delta_\mu\eta)$$

and Δ_ν is as in (4.4). Note that

$$\mathbf{q}_{b_\nu}(\Delta_\nu\xi) = \|\xi\|,$$

hence F is supported in the product $K \times K$, where

$$K = \{\xi \in \mathfrak{g}^\star : \|\xi\| < 1\}.$$

By the Sobolev inequality (10.1),

$$|\mathbf{U}(f)(\xi)| \le \max_{|\alpha|+|\beta|\le N+1} \|D_\xi^\alpha D_\eta^\beta F\|_{L^2(\mathfrak{g}^\star\times\mathfrak{g}^\star)},$$

where

$$\|D_\xi^\alpha D_\eta^\beta F\|^2_{L^2(\mathfrak{g}^\star\times\mathfrak{g}^\star)} = \int_K\int_K |D_\xi^\alpha D_\eta^\beta F(\xi,\eta)|^2 d\xi d\eta$$

$$= \int_K\int_K \mathbf{q}_\alpha(b_\nu)^2\mathbf{q}_\beta(b_\mu)^2|D_\xi^\alpha D_\eta^\beta f(b_\nu+\Delta_\nu\xi, b_\mu+\Delta_\mu\eta)|^2 d\xi d\eta.$$

Recall that, by (3.10),

$$\mathbf{q}_\alpha(b_\nu)\approx\mathbf{q}_\alpha(b_\nu+\Delta_\nu\xi),$$

for $b_\nu+\Delta_\nu\xi\in B_\nu$. Thus,

$$\|D_\xi^\alpha D_\eta^\beta F\|^2_{L^2(\mathfrak{g}^\star\times\mathfrak{g}^\star)} \lesssim \int_K\int_K |f|^{\mathbf{q}\oplus\mathbf{q}}_{N+1}(b_\nu+\Delta_\nu\xi, b_\mu+\Delta_\mu\eta)^2 d\xi d\eta$$

$$\lesssim \Big(|f|^{\mathbf{q}\oplus\mathbf{q}}_{N+1}\Big)^2.$$

□

Let

$$d_{\nu,k}(\xi) = 1+\sum_{j\succeq k}\frac{|\xi_j-(b_\nu)_j|}{q_j(b_\nu)}, \qquad \xi\in X^\star.$$

By Corollary 3.11, there exists $R>0$ such that, for every $k\in\mathcal{N}$,

$$\left(\frac{\widetilde{q}_k(\xi)}{\widetilde{q}_k(b_\nu)}\right)^{\pm1}\lesssim d_{\nu,k}(\xi)^R. \tag{6.2}$$

Recall that the functions $\widetilde{q}_k$ have been defined in Corollary 3.11.

Proposition 6.3. *For every $L\in\boldsymbol{N}$, there exists $s_0>0$ such that, for all ν,μ,*

$$|\mathbf{U}f(\xi)|\lesssim d_\nu(\xi)^{-L}d_\mu(\xi)^{-L}|f|^{\mathbf{q}\oplus\mathbf{q}}_{s_0}, \qquad \xi\in\mathfrak{g}^\star,$$

if $f\in C_c^\infty(B_\nu\times B_\mu)$.

Proof. For the sake of the proof we refine our claim:

For every $L\in\boldsymbol{N}$ and every $1\le k\le N$, there exists $s_k>0$ such that, for all ν,μ,

$$|\mathbf{U}f(\xi)|\lesssim d_{\nu,k}(\xi)^{-L}d_{\mu,k}(\xi)^{-L}|f|^{\mathbf{q}\oplus\mathbf{q}}_{s_k}. \tag{6.4}$$

Once we prove (6.4) for all minimal $k\in\mathcal{N}$, our claim will follow. We proceed by induction starting with maximal k. If k is maximal in $\mathcal{N}$, then, by (5.13),

$$\frac{\xi_k-(b_\nu)_k}{q_k(b_\nu)^{p_k}}\mathbf{U}f(\xi) = \mathbf{U}\Big(\frac{T_k^1-(b_\nu)_k}{q_k(b_\nu)^{p_k}}f\Big)(\xi),$$

where

$$\left|\frac{T_k^1-(b_\nu)_k}{q_k(b_\nu)^{p_k}}f\right|\le|f|, \qquad f\in C_c^\infty(B_\nu\times B_\mu),$$

so our claim is reduced to that of Lemma 6.1. Otherwise, assume that, for any $j\succ k$, any $L_1,L_2\in\boldsymbol{N}$, and some $s>0$,

$$|\mathbf{U}f(\xi)|\lesssim C(\xi)|f|^{\mathbf{q}\oplus\mathbf{q}}_s, \tag{6.5}$$

where

$$C(\xi)=\left(1+\frac{|\xi_k-(b_\nu)_k|}{q_k(b_\nu)}\right)^{-M_1}\left(1+\frac{|\xi_k-(b_\mu)_k|}{q_k(b_\mu)}\right)^{-M_2}d_{\nu,j}(\xi)^{-L_1}d_{\mu,j}(\xi)^{-L_2}.$$

The inductive step consists in showing that (6.5) implies that, for every $j\succ k$, there exists s' such that

$$\frac{|\xi_k-(b_\nu)_k|}{q_k(b_\nu)}|\mathbf{U}f(\xi)|\lesssim C_1(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}} \tag{6.6}$$

and

$$\frac{|\xi_k-(b_\mu)_k|}{q_k(b_\mu)}|\mathbf{U}f(\xi)|\lesssim C_1(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}}, \tag{6.7}$$

where

$$C_1(\xi)=d_{\nu,j}(\xi)^{2R}d_{\mu,j}(\xi)^{2R}C(\xi),$$

for some $m_1,R>0$. Since the cases (6.6) and (6.7) are almost identical, let us only consider the first one.

By (5.13),

$$\frac{\xi_k-(b_\nu)_k}{q_k(b_\nu)^{p_k}}\mathbf{U}f(\xi)=\mathbf{U}\Big(\frac{T_k^1-(b_\nu)_k}{q_k(b_\nu)^{p_k}}f\Big)(\xi) \tag{6.8}$$

$$-\sum_{j\succ k}\frac{\xi_j}{q_k(b_\nu)^{p_k}}\mathbf{U}\Big(\mathcal{D}^1_{j,k}f\Big)(\xi) \tag{6.9}$$

$$=\mathbf{U}(f_{\nu,k})(\xi)-\sum_{j\succ k}\frac{\xi_j}{q_k(b_\nu)^{p_k}}\mathbf{U}(\mathcal{D}^1_{j,k}f)(\xi). \tag{6.10}$$

Note that, by (6.5),

$$|\mathbf{U}f_{\nu,k}(\xi)|\lesssim C(\xi)\left|\frac{T_k-(b_\nu)_k}{q_k(b_\nu)^{p_k}}f\right|_{s'}^{\mathbf{q}\oplus\mathbf{q}}\lesssim C(\xi)|f|_s^{\mathbf{q}\oplus\mathbf{q}}.$$

To prove (6.6), it is, therefore, sufficient to estimate each of the terms

$$U_{k,j}(\xi)=\frac{\xi_j}{q_k(b_\nu)^{p_k}}\mathbf{U}(\mathcal{D}^1_{k,j}f)(\xi).$$

Note that, by (5.16),

$$\mathcal{D}^1_{j,k}=\sum_{p(\alpha)+p(\beta)=p_j-p_k}c_{\alpha\beta}D_\xi^\alpha D_\eta^\beta.$$

Therefore, by (6.5), there exists $s'>0$ such that

$$\begin{aligned}|\mathbf{U}(\mathcal{D}^1_{j,k}f)(\xi)|&\lesssim|\mathcal{D}^1_{j,k}f|_s^{\mathbf{q}\oplus\mathbf{q}}\\&\lesssim C(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}}\sum_{p(\alpha)+p(\beta)=p_j-p_k}\mathbf{q}_\alpha(b_\nu)^{-1}\mathbf{q}_\beta(b_\mu)^{-1},\end{aligned}$$

where $\alpha_r=\beta_r=0$ unless $r\prec j$. Therefore,

$$\mathbf{q}_\alpha(b_\nu)^{-1}\lesssim\widetilde{q}_j(b_\nu)^{-p(\alpha)},\qquad\mathbf{q}_\beta(b_\mu)^{-1}\lesssim\widetilde{q}_j(b_\mu)^{-p(\beta)}.$$

By (6.2),

$$|\mathbf{U}(\mathcal{D}^1_{j,k}f)(\xi)|\lesssim C(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}}\widetilde{q}_j(\xi)^{p_k-p_j}d_{\nu,j}(\xi)^{R/2}d_{\mu,j}(\xi)^R,$$

for some $R > 0$, so that

$$\begin{aligned}|U_{k,j}(\xi)| &\lesssim C(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}} \frac{|\xi_j|}{q_k(b_\nu)^{p_k}} \frac{\widetilde{q}_j(\xi)^{p_k}}{\widetilde{q}_j(\xi)^{p_j}} d_{\nu,j}(\xi)^{R/2} d_{\mu,j}(\xi)^R \\ &= C(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}} \frac{|\xi_j|}{\widetilde{q}_j(\xi)^{p_j}} \frac{\widetilde{q}_j(\xi)^{p_k}}{q_k(\xi)^{p_k}} d_{\nu,j}(\xi)^R d_{\mu,j}(\xi)^R.\end{aligned}$$

Now,

$$|\xi_j| \lesssim \widetilde{q}_j(\xi)^{p_j}, \qquad \widetilde{q}_j(\xi) \lesssim q_k(\xi),$$

hence

$$|U_{k,j}(\xi)| \lesssim C_1(\xi)|f|_{s'}^{\mathbf{q}\oplus\mathbf{q}}.$$

The proof is complete. □

Corollary 6.11. *Let $\boldsymbol{m}_1, \boldsymbol{m}_2$ be $\mathbf{q}$-weights on $\mathfrak{g}^\star$ and let $\boldsymbol{m} = \boldsymbol{m}_1 \otimes \boldsymbol{m}_2$. Let $s_0 > 0$ be as in Proposition 6.3. For every $s > 0$ and every $L \in \boldsymbol{N}$,*

$$|\mathbf{U}f|_s^{\boldsymbol{m}_1\boldsymbol{m}_2}(\xi) \lesssim |f|_{s_0+s}^{\boldsymbol{m}} d_\nu(\xi)^{-L} d_\mu(\xi)^{-L} \tag{6.12}$$

if $f \in C_c^\infty(B_\nu \times B_\mu)$. The estimate is uniform in ν, μ.

Proof. Let C be a multiindex with $p(C) \leq s$. Then, by Lemma 5.15,

$$|D^C \mathbf{U}f(\xi)| = |\mathbf{U}(\mathcal{D}^C f)(\xi)| \lesssim \sum_{\substack{p(A)+p(B)\leq s \\ \mathbf{q}^C \leq \mathbf{q}^A\mathbf{q}^B}} |\mathbf{U}(D_1^A D_2^B f)(\xi)|.$$

By Proposition 6.3,

$$\begin{aligned}|\mathbf{U}(D_1^A D_2^B f)(\xi)| &\lesssim d_\nu(\xi)^{-L} d_\mu(\xi)^{-L} |D_1^A D_2^B f|_{s_0}^{\mathbf{q}\oplus\mathbf{q}} \\ &\lesssim d_\nu(\xi)^{R-L} d_\mu(\xi)^{R-L} |f|_{s_0+s}^{\boldsymbol{m},\mathbf{q}\oplus\mathbf{q}} \boldsymbol{m}_1(\xi)\boldsymbol{m}_2(\xi)\mathbf{q}_A(\xi)^{-1}\mathbf{q}_B(\xi)^{-1}, \\ &\lesssim d_\nu(\xi)^{R-L} d_\mu(\xi)^{R-L} |f|_{s_0+s}^{\boldsymbol{m},\mathbf{q}\oplus\mathbf{q}} \boldsymbol{m}_1(\xi)\boldsymbol{m}_2(\xi)\mathbf{q}_C(\xi)^{-1},\end{aligned}$$

which implies

$$|\mathbf{U}f|_s^{\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q}}(\xi) \lesssim |f|_{s_0+s}^{\boldsymbol{m},\mathbf{q}\oplus\mathbf{q}} d_\nu(\xi)^{R-L} d_\mu(\xi)^{R-L}.$$

□

Recall that the numbers T and $d_{\nu\mu}$ have been defined respectively in Proposition 3.12 and just before Proposition 4.6.

Corollary 6.13. *For every $L > 0$, there exists $s > 0$ such that*

$$\|\mathbf{U}(f)\|_{A(\mathfrak{g}^\star)} \lesssim |f|_s^{\mathbf{q}\oplus\mathbf{q}} d_{\nu\mu}^{-L}, \qquad f \in C_c^\infty(B_\nu \times B_\mu),$$

uniformly in ν, μ.

Proof. Let

$$F(\xi) = \mathbf{U}f(\Delta_\nu \xi), \qquad \xi \in X^\star, \quad \nu \in \boldsymbol{N},$$

where Δ_ν is as in (4.4).Then,

$$\|\mathbf{U}f\|_{A(X^\star)} = \|F\|_{A(X^\star)},$$

and, by (10.1),

$$\|\mathbf{U}f\|_{A(X^\star)} \lesssim \max_{|\alpha|\leq N/2+1} \|D^\alpha F\|_{L^2(X^\star)},$$

where

$$\begin{aligned}|D^\alpha F(\xi)| &= \mathbf{q}^\alpha(b_\nu)|(D^\alpha \mathbf{U}f)(\Delta_\nu\xi)| \\ &\lesssim \Big(1+\mathbf{q}_{b_\nu}(b_\nu-\Delta_\nu\xi)\Big)^{p(\alpha)T}\mathbf{q}^\alpha(\Delta_\nu\xi)|(D^\alpha\mathbf{U}f)(\Delta_\nu\xi)| \\ &\lesssim d_\nu(\Delta_\nu\xi)^R|\mathbf{U}f|_r^{\mathbf{q}}(\Delta_\nu\xi),\end{aligned}$$

for sufficiently large $R, r>0$. Let

$$L_1 = L(2T+1)+\frac{N+1}{2}+R, \qquad L_2 = L(2T+1).$$

By Corollary 6.11,

$$\begin{aligned}|D^\alpha F(\xi)| &\lesssim |f|_{s_0+r}^{\mathbf{q}\oplus\mathbf{q}} d_\nu(\Delta_\nu\xi)^{-L_1+R}d_\mu(\Delta_\nu\xi)^{-L_2} \\ &\lesssim |f|_{s_0+r}^{\mathbf{q}\oplus\mathbf{q}} d_\nu(\Delta_\nu\xi)^{-L(2T+1)}d_\mu(\Delta_\nu\xi)^{-L(2T+1)}d_\nu(\Delta_\nu\xi)^{-\frac{N+1}{2}}.\end{aligned}$$

By Corollary 4.6,

$$|D^\alpha F(\xi)| \lesssim |f|_{s_0+r}^{\mathbf{q}\oplus\mathbf{q}} d_{\nu\mu}^{-L}\left(1+\sum_{k=1}^N\left|\xi_k-\frac{(b_\nu)_k}{q_k(b_\nu)^{p_k}}\right|\right)^{-\frac{N+1}{2}},$$

so that, finally,

$$\|\mathbf{U}f\|_{A(X^\star)} \lesssim \sum_{|\alpha|\le N/2+1}\|D^\alpha F\|_{L^2(X^\star)} \lesssim |f|_{s_0+r}^{\mathbf{q}\oplus\mathbf{q}} d_{\nu\mu}^{-L}.$$

□

7. Twisted multiplication and L^2-multipliers

Let φ_ν be the partition of unity of Proposition 4.1. For a function $f\in\mathcal{S}(\mathfrak{g}^\star\times\mathfrak{g}^\star)$, let

$$f_{\nu,\mu}(\xi,\eta) = \varphi_\nu(\xi)\varphi_\mu(\eta)f(\xi,\eta), \qquad \nu,\mu\in\boldsymbol{N}.$$

Recall that the weak convergence of linear and bilinear mappings has been defined in Section 3 (see (3.15) and below).

Theorem 7.1. *Let $\mathbf{g}\in\mathcal{G}$. Let $\boldsymbol{m}(\xi,\eta)=\boldsymbol{m}_1(\xi)\boldsymbol{m}_2(\eta)$, where $\boldsymbol{m}_1,\boldsymbol{m}_2$ are weights on $\mathfrak{g}^\star$. The linear mapping $\mathbf{U}$ defined on the weakly dense subspace $\mathcal{S}(\mathfrak{g}^\star\times\mathfrak{g}^\star)$ of $S(\boldsymbol{m},\mathbf{g}\oplus\mathbf{g})$ with values in $\mathcal{S}(\mathfrak{g}^\star)\subseteq S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{g})$ is weakly continuous. Consequently, it has a unique extension $\widetilde{\mathbf{U}}$ to a weakly continuous linear mapping from $S(\boldsymbol{m},\mathbf{g}\oplus\mathbf{g})$ into $S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{g})$.*

Proof. We begin with $\mathbf{g}=\mathbf{q}$. Let $f_n\in\mathcal{S}(\mathfrak{g}^\star\times\mathfrak{g}^\star)$ be weakly convergent in $S(\boldsymbol{m},\mathbf{q}\oplus\mathbf{q})$ to 0. Then,

$$\mathbf{U}f_n = \sum_{\mu,\nu}\mathbf{U}(f_n)_{\nu,\mu} \tag{7.2}$$

in $\mathcal{S}(\mathfrak{g}^\star)$. By (6.12), for every s, such that

$$\begin{aligned}|\mathbf{U}f_n|_s^{\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q}}(\xi) &\le \sum_{\mu,\nu}|\mathbf{U}(f_n)_{\nu,\mu}|_s^{\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q}}(\xi) \\ &\lesssim \sum_{\nu,\mu}|(f_n)_{\nu,\mu}|_{s_0+s}^{\boldsymbol{m},\mathbf{q}\oplus\mathbf{q}}d_\nu(\xi)^{-L}d_\mu(\xi)^{-L}, \qquad (7.3)\end{aligned}$$

where L is so large that the series is convergent (see (4.2)). Thus,

$$|\mathbf{U}f_n|_s^{\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q}} \lesssim |f_n|_{s_0+s}^{\boldsymbol{m},\mathbf{q}\oplus\mathbf{q}}, \tag{7.4}$$

which shows that the sequence $\mathbf{U}f_n$ is bounded in $S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q})$. The series (7.2) is absolutely pointwise convergent. Formula (7.3) with $m=0$ implies that it is convergent uniformly in n. Therefore, the sequence $\mathbf{U}f_n$ is pointwise convergent to 0. Being bounded, it is weakly convergent in $S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{q})$. This completes the first part of the proof.

Now, let $\mathbf{g}\in\mathcal{G}$. Let $f_n\in\mathcal{S}(\mathfrak{g}^\star\times\mathfrak{g}^\star)$ be weakly convergent to 0 in $S(\boldsymbol{m},\mathbf{g}\oplus\mathbf{g})$. Since $\mathbf{g}\le\mathbf{q}$, the sequence is weakly convergent in $S(\boldsymbol{m},\mathbf{q}\oplus\mathbf{q})$. Therefore, by the first part of the proof, $\mathbf{U}f_n$ is convergent to 0 in the C^∞-toplology. It remains to show that it is bounded in $S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{g})$.

Let α be a multiindex of length $p(\alpha)\le s$. By (5.14) and Lemma 5.15,

$$\begin{aligned}\boldsymbol{m}_1^{-1}\boldsymbol{m}_2^{-1}\mathbf{g}^\alpha|D^\alpha\mathbf{U}f| &= \boldsymbol{m}_1^{-1}\boldsymbol{m}_2^{-1}\mathbf{g}^\alpha|\mathbf{U}(\mathcal{D}^\alpha f)|\\ &\lesssim \sum_{A,B}(\boldsymbol{m}_1^{-1}\mathbf{g}^A)(\boldsymbol{m}_2^{-1}\mathbf{g}^B)|\mathbf{U}(D_\xi^A D_\eta^B)f| = \sum_{A,B}\boldsymbol{n}_A^{-1}\boldsymbol{n}_B^{-1}|\mathbf{U}(D_\xi^A D_\eta^B)f|\\ &\lesssim \sum_{A,B}|\mathbf{U}(D_\xi^A D_\eta^B f)|_0^{\boldsymbol{n}_A\boldsymbol{n}_B,\mathbf{q}},\end{aligned}$$

where the summation is extended over A,B such that $p(A)+p(B)=p(\alpha)$, and $\boldsymbol{n}_A=\mathbf{g}^{-A}\boldsymbol{m}_1$, $\boldsymbol{n}_B=\mathbf{g}^{-B}\boldsymbol{m}_2$. By (7.4),

$$\begin{aligned}|\mathbf{U}(D_\xi^A D_\eta^B f)|_0^{\boldsymbol{n}_A\boldsymbol{n}_B,\mathbf{q}} &\lesssim |D_\xi^A D_\eta^B f|_{s_0}^{\boldsymbol{n}_A\otimes\boldsymbol{n}_B,\mathbf{q}\oplus\mathbf{q}}\\ &= \max_{p(a)+p(b)\le s_0}\sup_{\xi,\eta}\boldsymbol{m}^{-1}(\mathbf{g}^A\otimes\mathbf{g}^B)(\mathbf{q}^a\otimes\mathbf{q}^b)|D_\xi^a D_\eta^b D_\xi^A D_\eta^B f|\\ &\lesssim \max_{p(a)+p(b)\le s_0}\sup_{\xi,\eta}\boldsymbol{m}^{-1}(\mathbf{g}^{A+a}\otimes\mathbf{g}^{B+b})|D_\xi^{A+a}D_\eta^{B+b}f|\\ &\le |f|_{s_0+s}^{\boldsymbol{m},\mathbf{g}\oplus\mathbf{g}}.\end{aligned}$$

We have taken advantage of the fact that $\mathbf{q}^\beta\le\mathbf{g}^\beta$. Thus,

$$|\mathbf{U}f|_s^{\mathbf{g},\boldsymbol{m}_1\boldsymbol{m}_2}\lesssim|f|_{s_0+s}^{\boldsymbol{m},\mathbf{g}\oplus\mathbf{g}},$$

which shows that the sequence $\mathbf{U}f_n$ is bounded in $S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{g})$.

□

Remark 7.5. The extension $\widetilde{\mathbf{U}}$ will be still denoted by $\mathbf{U}$.

The twisted product of $f,g\in\mathcal{S}(\mathfrak{g}^\star)$ is defined by

$$f\#g(\xi)=(f^\vee\star g^\vee)^\wedge(\xi),\qquad \xi\in\mathfrak{g}^\star. \tag{7.6}$$

It is checked directly that

$$f\#g(\xi)=\mathbf{U}(f\otimes g)(\xi),\qquad f,g\in\mathcal{S}(\mathfrak{g}^\star). \tag{7.7}$$

As an immediate application of Theorem 7.1 and (7.7) we obtain the following extension of the twisted product.

Corollary 7.8. *Let $\mathbf{g}\in\mathcal{G}$. Let $\boldsymbol{m}_1,\boldsymbol{m}_2$ be $\mathbf{g}$-weights on $\mathfrak{g}^\star$. The twisted product $(f,g)\mapsto f\#g$ defined for Schwartz functions on $\mathfrak{g}^\star$ extends uniquely to a weakly continuous bilinear mapping*

$$S(\boldsymbol{m}_1,\mathbf{g})\times S(\boldsymbol{m}_2,\mathbf{g})\ni(f,g)\mapsto f\#g\in S(\boldsymbol{m}_1\boldsymbol{m}_2,\mathbf{g}),$$

denoted by the same symbol $\#$.

Proof. The mapping

$$S(\boldsymbol{m}_1, \mathbf{g}) \times S(\boldsymbol{m}_2, \mathbf{g}) \ni (f, g) \mapsto f \otimes g \in S(\boldsymbol{m}_1 \otimes \boldsymbol{m}_2, \mathbf{g} \oplus \mathbf{g})$$

is obviously weakly continuous. Thus, our claim follows from Theorem 7.1. □

Definition 7.9. Let $\mathcal{N}_{\max}$ be the set of maximal elements in $\mathcal{N}$. Denote by $\mathfrak{z} = \langle e_k \rangle_{k \in \mathcal{N}_{\max}}$ the linear subspace of $\mathfrak{g}$ spanned by the vectors e_k such that $k \in \mathcal{N}_{\max}$. By Lemma 5.7, $\mathfrak{z}$ is a subspace of the centre of $\mathfrak{g}$. Let $\mathcal{N}_0 = \mathcal{N} \setminus \mathcal{N}_{\max}$. Let $\mathfrak{g}_0 = \langle e_k \rangle_{k \in \mathcal{N}_0}$. Let $\mathfrak{z}^\star = \langle e_k^\star \rangle_{k \in \mathcal{N}_{\max}}$ and $\mathfrak{g}_0^\star = \langle e_k^\star \rangle_{k \in \mathcal{N}_0}$.

Recall that $\mathcal{M}(\mathbf{q})$ denotes the set of all $\mathbf{q}$-weights. Let

$$S(\mathbf{q}) = \bigcup_{\boldsymbol{m} \in \mathcal{M}(\mathbf{q})} S(\boldsymbol{m}, \mathbf{q}).$$

Note that, for every $\mathbf{g} \in \mathcal{G}$ and every weight $\boldsymbol{m}$, $S(\boldsymbol{m}, \mathbf{g}) \subseteq S(\mathbf{q})$. A sequence a_n in $S(\mathbf{q})$ is said to be weakly convergent if it is weakly convergent in one of the spaces $S(\boldsymbol{m}, \mathbf{g})$.

Assume that $M^\wedge \in C^\infty(\mathfrak{g}^\star)$ is symmetric, depends only on the $\mathfrak{z}^\star$-variable, and

$$M^\wedge(\xi) = \begin{cases} 0, & R(\xi) \leq 1/2, \\ 1, & R(\xi) \geq 1, \end{cases}$$

where $R(\xi) = \min_{k \in \mathcal{N}_{\max}} |\xi_k|^{1/p_k}$. Then, M is a central $\mathcal{S}$-convolver (see (10.8) and (10.9)) and

$$(M \star K)^\wedge = M^\wedge K^\wedge,$$

for every $K \in \mathcal{S}'(\mathfrak{g})$. Let also

$$\mathcal{S}_0(\mathfrak{g}) = \{f \in \mathcal{S}(\mathfrak{g}) : \exists_{\delta > 0}\ \widehat{f}(\xi) = 0 \text{ for } R(\xi) < \delta\}. \tag{7.10}$$

This space is dense in $L^2(\mathfrak{g})$.

Recall the Cotlar-Stein lemma (see Stein [19], Chapter VII, 2.1 and 5.3).

Lemma 7.11. *Let T_ν be a sequence of bounded linear operators on a Hilbert space. Suppose there exists a constant $C > 0$ such that*

$$\sum_\nu \|T_\nu T_\mu^\star\|^{1/2} \leq C, \qquad \sum_\nu \|T_\nu^\star T_\mu\|^{1/2} \leq C.$$

Then, the series $\sum_\nu T_\nu$ is strongly convergent.

Let

$$H^\star = \{\xi \in \mathfrak{g}^\star : \exists_{k \in \mathcal{N}_{\max}}\ \xi_k = 0\}.$$

Theorem 7.12. *Let $K \in \mathcal{S}'(\mathfrak{g})$ be such that $\widehat{K}$ is locally integrable, smooth away from $H^\star$, and satisfies the estimates*

$$|D^\alpha \widehat{K}(\xi)| \leq C_\alpha \prod_{k \in \mathcal{N}_0} N_k(\xi)^{-p_k \alpha_k}, \qquad \xi \notin H^\star,$$

where $N_k(\xi) = \sum_{j \succ k} |\xi_j|^{1/p_j}$ with the ordering $\prec$ of Definition 5.5. Then,

$$\|f \star K\|_{L^2(\mathfrak{g})} \lesssim C_K \|f\|_{L^2(\mathfrak{g})}, \qquad f \in C_c^\infty(\mathfrak{g}),$$

where C_K depends on a finite number of constants C_α. Thus, the operator $f \mapsto f \star K$ extends to a bounded operator on $L^2(\mathfrak{g})$.

Proof. Let us assume first that $\widehat{K} \in S(\mathbf{1}, \mathbf{q})$. This part of the proof is based on the Cotlar-Stein lemma. Let $a = \widehat{K}$. We write $a = \sum_\nu a_\nu$, where $a_\nu(\xi) = a(\xi)\varphi_\nu(\xi)$ and $\{\varphi_\nu\}$ is the partition of unity of Proposition 4.1 for the metric $\mathbf{q}$. Let $Tf = f\#a$ and $T_\nu f = f\#a_\nu$. By Corollary 6.13, there exists $s > 0$ such that

$$\|T_\nu T_\mu^\star\| \le \|(a_\nu \# \bar{a}_\mu)^\vee\|_{L^1(\mathfrak{g})} \lesssim (|a|_s^{\mathfrak{q}})^2 d_{\nu\mu}^{-2N-1}, \tag{7.13}$$

which implies

$$\sum_\nu \|T_\nu T_\mu^\star\|^{1/2} \le C,$$

for some $C > 0$. By replacing a by $\bar{a}$, we get analogous estimate for $\|T_\nu^\star T_\mu\|^{1/2}$.

Our argument so far shows that, by the Cotlar-Stein lemma, the series is convergent in the strong sense. Thus, the operator $\sum_\nu T_\nu$ is bounded. Since

$$T_\nu f = f \star K_\nu, \qquad \widehat{K_\nu} = a_\nu,$$

and, obviously, $K = \sum_\nu K_\nu$ in the sense of distributions, we get

$$\sum_\nu T_\nu f = Tf = f \star K,$$

for $f \in \mathcal{S}(\mathfrak{g})$. The norm of T is bounded by $C_K = C$.

We turn to the general case. It is not hard to see that $M \star K$ has Fourier transform in $S(\mathbf{1}, \mathbf{q})$. Note that, for every $f \in S_0(\mathfrak{g})$, there exists $t > 0$ such that

$$f_t = f_t \star M, \qquad f_t(x) = t^{-Q} f(\delta_{t^{-1}} x).$$

Therefore,

$$\begin{aligned}\|f \star K\|_{L^2(\mathfrak{g})} &= t^{Q/2}\|f_t \star K_t\|_{L^2(\mathfrak{g})} \\ &= t^{Q/2}\|(f_t \star M) \star K_t\|_{L^2(\mathfrak{g})} = t^{Q/2}\|f_t \star (M \star K_t)\|_{L^2(\mathfrak{g})},\end{aligned}$$

where $M \star K_t$ satisfies the estimates of Theorem 7.12 with constants $\lesssim C_\alpha$ independently of t. Consequently,

$$\|f \star K\|_{L^2(\mathfrak{g})} \lesssim C t^{Q/2} \|f_t\|_{L^2(\mathfrak{g})} = C\|f\|_{L^2(\mathfrak{g})},$$

which proves our case. □

Remark 7.14. When $\mathfrak{g}$ is a two step nilpotent Lie algebra, then Theorem 7.12 reduces to the simplest version of the Calderón-Vaiilancourt theorem for the class $S^0_{0,0}$ of symbols. (See Stein [19], chapter VII, 2.4.)

8. Convolution of $\mathfrak{N}$-kernels

We have defined the twisted multiplication by (7.6) for $f, g \in \mathcal{S}(\mathfrak{g}^\star)$, and then extended it to the elements of our symbol classes. On the other hand the convolution on $\mathfrak{g}$ can be also extended to various types of distributions. The most obvious case is when one of the distributions has compact support. A general definition of convolvable distributions due to Chevalley [1] is given in Definition 10.3.

Proposition 8.1. i) *If* $a \in S(\mathbf{q})$ *and* $f \in \mathcal{S}(\mathfrak{g}^\star)$, *then*

$$a\#f, f\#a \in \mathcal{S}(\mathfrak{g}^\star).$$

ii) *If* $a, b \in S(\mathbf{q})$ *and* $f \in \mathcal{S}(\mathfrak{g}^\star)$, *then*

$$\langle a\#b, f\rangle = \langle a, f\#\widetilde{b}\rangle = \langle b, \widetilde{a}\#f\rangle.$$

iii) *Let* $\widehat{T} \in S(\mathbf{q})$ *and* $f \in \mathcal{S}(\mathfrak{g})$. *Then,*

$$(T \star f)^\wedge = \widehat{T}\#\widehat{f}, \qquad (f \star T)^\wedge = \widehat{f}\#\widehat{T}.$$

iv) *Let* $\widehat{S}, \widehat{T} \in S(\mathbf{q})$. *Then,*

$$(S \star T)^\wedge = \widehat{S}\#\widehat{T}.$$

Remark 8.2. By i) and iii), every distribution T such that $\widehat{T} \in S(\mathbf{q})$ is an $\mathcal{S}$-convolver (see (10.8)). Therefore, by (10.9), iv) makes sense.

Proof of (8.1). i) Let $a \in S(\boldsymbol{m}, \mathbf{q})$ for a $\mathbf{q}$-weight $\boldsymbol{m}$. For every $m > 0$, $f \in S(\boldsymbol{m}^{-1}(1+\rho)^{-m}, \mathbf{q})$. By Corollary 7.8, $a\#f, f\#a \in S((1+\rho)^{-m}, \mathbf{q})$. Since m is arbitrary $a\#f, f\#a \in \mathcal{S}(\mathfrak{g}^\star)$.

ii) This is obvious by weak approximation.

iii) We have

$$\langle (T \star f)^\wedge, \varphi\rangle = \langle T \star f, \varphi^\vee\rangle = \langle T, \varphi^\vee \star \widetilde{f}\rangle,$$

whence

$$\langle (T \star f)^\wedge, \varphi\rangle = \langle \widehat{T}, \varphi\#(\widehat{f})^\sim\rangle = \langle \widehat{T}\#\widehat{f}, \varphi\rangle.$$

iv) Similarly,

$$\langle (S \star T)^\wedge, \varphi\rangle = \langle S \star T, \varphi^\vee\rangle = \langle S, \varphi^\vee \star \widetilde{T}\rangle,$$

whence, by iii) and ii),

$$\langle (S \star T)^\wedge, \varphi\rangle = \langle \widehat{S}, \varphi\#\widetilde{\widehat{T}}\rangle = \langle \widehat{S}\#\widehat{T}, \varphi\rangle.$$

□

The concept of an $\mathfrak{N}$-kernel we are about to introduce is independent of the group structure. To underscore this fact we return for a while to the setting of the vector space X and its dual $X^\star$.

Let $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$ be a closed filtration in $\mathcal{N}$ (cf. Section 3). Recall that a filtration determines a set of partial homogeneous norms

$$N_k(\xi) = \sum_{j\in\mathcal{N}_k} |\xi_j|^{1/p_j}, \qquad k \in \mathcal{N}, \xi \in X^\star,$$

and the metric

$$\mathbf{g}_\xi(\eta) = \sum_{k\in\mathcal{N}} \frac{|\eta_k|}{(1 + N_k(\xi))^{p_k}}, \qquad \xi, \eta \in X^\star,$$

which is in the class $\mathcal{G}$. If $M \in \mathcal{A}_{\boldsymbol{R}}$, then, by Proposition 3.18,

$$\boldsymbol{m}_M(\xi) = \prod_{k\in\mathcal{N}} (1 + N_k(\xi))^{m_k}$$

is a weight.

We say that a finite set $\emptyset \neq J \subseteq \mathcal{A}_{\boldsymbol{R}}$ of multiindices $M = (m_1, m_2, \dots, m_N)$ is *admissible* if

$$\sum_{M \in I} m_j > -p_j/2, \qquad j \in \mathcal{N},$$

for every $\emptyset \neq I \subseteq J$. If a singleton $\{M\}$ is admissible, we just say that M is admissible.

Definition 8.3. For a closed filtration $\mathfrak{N}$ and an admissible multiindex $M = (m_1, m_2, \dots, m_N)$, we denote by $\boldsymbol{F}^M(\mathfrak{N})$ the class of all $K \in \mathcal{S}'(\mathfrak{g})$ such that $\widehat{K}$ is locally integrable, smooth away from $H^\star$ (see below), and satisfies

$$|D^\alpha \widehat{K}(\xi)| \lesssim \prod_{k \in \mathcal{N}} N_k(\xi)^{m_k - p_k \alpha_k}, \qquad \alpha \in \mathcal{A}_{\boldsymbol{N}}, \quad \xi \in \mathfrak{g}^\star \setminus H^\star. \tag{8.4}$$

Let also

$$\|K\|_{\boldsymbol{F}^M, j} = \max_{p(\alpha) \leq j} \sup_{\xi \in X^\star \setminus H^\star} \prod_{k \in \mathcal{N}} N_k(\xi)^{-m_k + p_k \alpha_k} |D^\alpha \widehat{K}(\xi)|.$$

Note that

$$\|K_t\|_{\boldsymbol{F}^M, j} = t^{\sigma(M)} \|K\|_{\boldsymbol{F}^M, j}, \qquad t > 0, \tag{8.5}$$

where

$$\langle K_t, f \rangle = \langle K, f \circ \delta_t \rangle, \qquad \sigma(M) = \sum_{k \in \mathcal{N}} m_k.$$

Remark 8.6. Recall that $H^\star$ is the set of all $\xi \in X^\star$ such that $\xi_k = 0$ for some $k \in \mathcal{N}_{\max}$. By (10.2), $\widehat{K}$ is in fact smooth and satisfies the estimates (8.4) outside the set

$$H^\star_{\mathfrak{N}} = \{\xi \in X^\star : \exists_{k \in \mathcal{N}} \ N_k(\xi) = 0\} \subseteq H^\star.$$

Remark 8.7. If $K \in \boldsymbol{F}^M(\mathfrak{N})$, then $\widehat{K}$ is locally square-integrable.

Remark 8.8. If $K, L \in \boldsymbol{F}^M(\mathfrak{N})$ and

$$\langle K, f \rangle = \langle L, f \rangle, \qquad f \in \mathcal{S}_0(X),$$

then $K = L$. The space $\mathcal{S}_0(X)$ has been defined by (7.10).

Remark 8.9. Let $\mathfrak{N}$ be a closed filtration. The classes $\boldsymbol{F}^M(\mathfrak{N})$ and $S^M(\mathfrak{N}) = S(\boldsymbol{m}_M, \mathbf{g}_{\mathfrak{N}})$ (see Proposition 3.18) are related in a formally similar way as the classes of flag kernels and truncated flag kernels in Nagel-Ricci-Stein-Wainger [16], Definition 6.20. Note, however, the difference. One relationship is expressed in terms of Fourier transforms, the other in terms of the kernels themselves.

Example 8.10. Assume that $N > 1$. Let

$$k_0 = 1 < k_1 < \dots < k_r < k_{r+1} = N + 1.$$

If $k_l \leq k < k_{l+1}$, let

$$\mathcal{N}_k = \{j \in \mathcal{N} : j \geq k_l\}.$$

This is a closed filtration. Note that the linear space generated by $\{e_j\}_{j \in \mathcal{N}_k}$ is an algebra. The class $\boldsymbol{F}^0(\mathfrak{N})$ is exactly the class of *flag kernels* of Nagel-Ricci-Stein [15] or Nagel-Ricci-Stein-Wainger [16] corresponding to the flag

$$\{0\} \subseteq \langle e_j \rangle_{j < k_1} \subseteq \ldots \subseteq \langle e_j \rangle_{j < k_l} \subseteq \ldots \subseteq \langle e_j \rangle_{j \leq N} = X.$$

In particular, when $r = 0$, we get $\mathcal{N}_k = \mathcal{N}$, for every k. Then, the flag kernels become Calderón-Zygmund kernels.

Example 8.11. Let $\mathfrak{N} = \{\mathcal{N}_k\}$, where, for every $k \in \mathcal{N}$,

$$\mathcal{N}_k = \{j \in \mathcal{N} : j \succeq k\}.$$

This is a closed filtration and $\boldsymbol{F}^0(\mathfrak{N})$ is the class of L^p-multiplier kernels of Theorem 9.1 below. The filtration $\mathfrak{N}$ is minimal in the sense that if $\mathfrak{M} = \{\mathcal{M}_k\}$ is a closed filtration, then $\mathcal{N}_k \subseteq \mathcal{M}_k$, for every $k \in \mathcal{N}$.

Definition 8.12. Let $\mathfrak{N}$ be a closed filtration and $M \in \mathcal{A}_{\boldsymbol{R}}$ an admissible multiindex. Let $\mathbf{g} = \mathbf{g}_{\mathfrak{N}}$ and $\boldsymbol{m} = \boldsymbol{m}_M$. A sequence $K_\nu \in \mathcal{S}'(\mathfrak{g})$ will be called an $(\mathfrak{N}, M)$-*approximate sequence* if it is convergent in $\mathcal{S}'(\mathfrak{g})$ and the sequence

$$a_\nu = \nu^{\sigma(M)}(K_\nu)^\wedge \circ \delta_{1/\nu} \in S(\boldsymbol{m}, \mathbf{g})$$

is bounded.

Lemma 8.13. *Let $K \in \boldsymbol{F}^M(\mathfrak{N})$, where $\mathfrak{N}$ is a closed filtration and M is admissible. Then there exists a $(\mathfrak{N}, M)$-approximate sequence K_ν convergent to K in $\mathcal{S}'(X)$.*

Proof. Let $X_k^\star$ be the linear subspace of $X^\star$ generated by $\{e_j^\star\}_{j\in\mathcal{N}_k}$. Denote by $\xi_{(k)} = (\xi_j)_{j\in\mathcal{N}_k}$ the variable in $X_k^\star$. Note that N_k is a homogenous norm in $X_k^\star$. Let $\psi_k \in C^\infty(\mathfrak{g}_k^\star)$ be equal to 0 for $N_k(\xi_{(k)}) \le 1/2$ and equal to 1 for $N_k(\xi_{(k)}) \ge 1$. For every $\nu \in \boldsymbol{N} \setminus \{0\}$, let

$$(K_\nu)^\wedge(\xi) = \psi_1(\delta_\nu\xi_{(1)})\psi_2(\delta_\nu\xi_{(2)}) \dots \psi_N(\delta_\nu\xi_{(N)})\widehat{K}(\xi), \qquad \xi \in X^\star.$$

Then, for every ν and every $k \in \mathcal{N}$, $N_k(\xi) \approx 1/\nu$ on the support of $\widehat{K}_\nu$. Therefore,

$$a_\nu = \nu^{\sigma(M)}(K_\nu)^\wedge \circ \delta_{1/\nu} \in S(\boldsymbol{m}, \mathbf{g})$$

and

$$|a_\nu|_s^{\boldsymbol{m},\mathbf{g}} \lesssim \|K\|_{\boldsymbol{F}^M,s}, \qquad s \in \boldsymbol{N}.$$

Now, $\widehat{K}$ and $(K_\nu)^\wedge$ are uniformly locally square-integrable, uniformly polynomially bounded, and $(K_\nu)^\wedge(\xi) \to \widehat{K}(\xi)$ pointwise almost everywhere. Therefore, $(K_\nu)^\wedge \to \widehat{K}$ in $\mathcal{S}'(X^\star)$ which is equivalent to $K_\nu \to K$ in $\mathcal{S}'(X)$. □

We return to the Lie algebra setting.

Proposition 8.14. *Let $M \in \mathcal{A}_{\boldsymbol{R}}$ be admissible and let $\mathfrak{N}$ be a closed filtration. Let K_ν be a $(\mathfrak{N}, M)$-approximate sequence convergent to $K \in \mathcal{S}'(\mathfrak{g})$. Then, $(K_\nu)^\wedge$ is convergent in $C^\infty(\mathfrak{g}^\star \setminus H^\star)$ to $\widehat{K}$ and $K \in \boldsymbol{F}^M(\mathfrak{N})$. Moreover, for every $\varphi \in \mathcal{S}(\mathfrak{g})$, $K \star \varphi \in L^2(\mathfrak{g})$ and $K_\nu \star \varphi \to K \star \varphi$ in $L^2(\mathfrak{g})$.*

Proof. We have

$$|D^\alpha \widehat{K}_\nu(\xi)| \lesssim \prod_{k\in\mathcal{N}} N_k(\xi)^{m_k - p_k\alpha_k}$$

uniformly in ν and $\xi \in \mathfrak{g}^\star \setminus H^\star$. Therefore, by the Ascoli theorem, $D^\alpha \widehat{K}_\nu$ converges uniformly, for every α, on every compact subset of $\mathfrak{g}^\star \setminus H^\star$. Since $\widehat{K}_\nu \to \widehat{K}$ in the sense of distributions, the C^∞-limit must be equal to $\widehat{K}$. Consequently, $K \in \boldsymbol{F}^M(\mathfrak{N})$.

For $M' = (m'_1, m'_2, \dots, m'_N) \in \mathcal{A}_{\boldsymbol{R}}$, we let

$$\boldsymbol{m}'(\xi) = \prod_{k\in\mathcal{N}} (1 + N_k(\xi))^{m'_k}, \qquad \xi \in \mathfrak{g}^\star.$$

Let $\varphi \in \mathcal{S}(\mathfrak{g})$. Then, for every $M' \in (-\infty, 0]^N$, $\widehat{\varphi} \in S(\boldsymbol{m}', \mathbf{g})$. By Corollary 7.8,

$$(K_\nu \star \varphi)^\wedge \in S(\boldsymbol{m}', \mathbf{g}).$$

Let $0 < \varepsilon < 1/2$. There exist multiindices M' with $m'_k \leq 0$ such that $m_k + m'_k = -p_k/2 + \varepsilon_k$, where $\varepsilon_k = \pm\varepsilon$. Thus,

$$|(K_\nu \star \varphi)^\wedge(\xi)| \lesssim \prod_{k\in\mathcal{N}} \gamma\Big(N_k(\xi)\Big)^\varepsilon N_k(\xi)^{-p_k/2}, \qquad \xi_k \neq 0,$$

where $\gamma(t) = \min(t, t^{-1})$, which shows that the sequence $F_\nu = (K_\nu \star \varphi)^\wedge$ is dominated by a square-integrable function. Furthermore, F_ν is convergent in $C^\infty(\mathfrak{g}^\star \setminus H^\star$ (cf. the first part of the proof), hence almost everywhere. By the Lebesgue theorem, we conclude that F_ν is convergent in $L^2(\mathfrak{g}^\star)$ to $(K \star \varphi)^\wedge$, which implies our assertion. □

Theorem 8.15. *Let $\mathfrak{N}$ be a closed filtration. Let $K \in \boldsymbol{F}^{M_1}(\mathfrak{N})$, $L \in \boldsymbol{F}^{M_2}(\mathfrak{N})$, where the set $\{M_1, M_2\}$ of multiindices is admissible. Then the distributions K and L are convolvable and $K \star L \in \boldsymbol{F}^{M_1+M_2}(\mathfrak{N})$.*

Proof. By Propositions 8.14 and 10.4, K and L are convolvable. Let K_ν and L_ν be the approximate sequences converging to K and L, respectively. Then, by Theorem 7.1, the sequence

$$\widehat{U}_\nu = \widehat{K}_\nu \# \widehat{L}_\nu \in S(\boldsymbol{m}_1\boldsymbol{m}_2, \mathbf{g})$$

is bounded. This implies that the distributions U_ν are equicontinuous in $\mathcal{S}'(\mathfrak{g})$. Once we show that U_ν is convergent to $K \star L$ in $\mathcal{S}'(\mathfrak{g})$, we shall be able to conclude that U_ν is an $(M_1 + M_2, \mathfrak{N})$-approximate sequence and $K \star L \in \boldsymbol{F}^{M_1+M_2}(\mathfrak{N})$.

Let $f, g \in \mathcal{S}(\mathfrak{g})$. By Proposition 10.4,

$$\langle U_\nu, f \star g\rangle = \int_{\mathfrak{g}} \widetilde{K}_\nu \star f(x) L_\nu \star \widetilde{g}(x)\,dx,$$

where, by Proposition 8.14, $\widetilde{K}_\nu \star f \to \widetilde{K} \star f$ and $L_\nu \star \widetilde{g} \to L \star \widetilde{g}$ in $L^2(\mathfrak{g})$. Therefore,

$$\langle K \star L, f \star g\rangle = \int_{\mathfrak{g}} \widetilde{K} \star f(x) L \star \widetilde{g}(x)\,dx = \lim_\nu \langle U_\nu, f \star g\rangle,$$

which shows that, in fact, $U_\nu \to K \star L$ in $\mathcal{S}'(\mathfrak{g})$. Since the distributions U_ν are equicontinuous, it is sufficient to test the convergence on functions of the form $f \star g$, where $f, g \in \mathcal{S}(\mathfrak{g})$.

□

Corollary 8.16. *Let $\mathfrak{N}$ be a closed filtration. Let $\{M_1, M_2, M_3\}$ be an admissible set of multiindices. If $K_j \in \boldsymbol{F}^{M_j}(\mathfrak{N})$, $1 \leq j \leq 3$, then*

$$K_1 \star (K_2 \star K_3) = (K_1 \star K_2) \star K_3.$$

Proof. By Theorem 8.15, all convolutions are legitimate. Let

$$K_L = K_1 \star (K_2 \star K_3), \qquad K_R = (K_1 \star K_2) \star K_3,$$

and $C_j f = K_j \star f$, for $1 \le j \le 3$. It is not hard to see that C_j maps $\mathcal{S}_0(\mathfrak{g})$ into $\mathcal{S}_0(\mathfrak{g})$, so

$$K_L \star f = K_R \star f, \qquad f \in \mathcal{S}_0(\mathfrak{g}).$$

Both sides are continuous functions, so

$$\langle K_L, f\rangle = \langle K_R, f\rangle, \qquad f \in \mathcal{S}_0(\mathfrak{g}),$$

which, by Remark 8.8, implies $K_L = K_R$. □

In our terminology a product kernel K is a distribution in $\boldsymbol{F}^0(\mathfrak{N}')$, where $\mathcal{N}'_k = \{k\}$. This filtration is associated with the usual linear order in $\mathcal{N}$.

Proposition 8.17. *Let $M \in \mathcal{A}_{\boldsymbol{R}}$ be admissible. Let K be a Schwartz function regarded as a kernel in $\boldsymbol{F}^M(\mathfrak{N}')$. Then, there exists $l > 0$ such that*

$$|K(x)| \lesssim \|K\|_{\boldsymbol{F}^M,l} \prod_{k=1}^{N} |x_k|^{-1-m_k/p_k}, \qquad x_k \neq 0,\ k \in \mathcal{N}. \tag{8.18}$$

Proof. The case of product kernels is dealt with in Theorem 2.1.11 of Nagel-Ricci-Stein [15]. The general case is very similar. □

Corollary 8.19. *Let $M \in \mathcal{A}_{\boldsymbol{R}}$ be admissible and let $\mathfrak{N}$ be a closed filtration. Let $K \in \boldsymbol{F}^M(\mathfrak{N})$ be a continuous function. Then, there exists $l > 0$ such that*

$$|K(x)| \lesssim \|K\|_{\boldsymbol{F}^M,l} \prod_{k=1}^{N} |x_k|^{-1-m_k/p_k}, \qquad x_k \neq 0.$$

Proof. By Lemma 8.13, there is no harm in assuming that $K \in \mathcal{S}(\mathfrak{g})$ as long as the estimates only depend on seminorms in $\boldsymbol{F}^M(\mathfrak{N})$. As in Proposition 8.17, we may regard K as a member of $\boldsymbol{F}^M(\mathfrak{N}')$ and get

$$|K(x)| \lesssim \|K\|_{\boldsymbol{F}^M(\mathfrak{N}'),l} \prod_{k=1}^{N} |x_k|^{-1-m_k/p_k}, \qquad x_k \neq 0,\ k \in \mathcal{N}.$$

Since the filtration $\mathfrak{N}$ is closed, $\|K\|_{\boldsymbol{F}^m(\mathfrak{N}'),l} \le \|K\|_{\boldsymbol{F}^m(\mathfrak{N}),l}$, which gives the desired estimate. □

Remark 8.20. One could have defined admissible multiindices M as satisfying the weaker constraint $m_k > -p_k$, for $k \in \mathcal{N}$. Thus we could have had a broader class of convolvable kernels. However, what has been achieved is sufficient for proving the multiplier theorem in the next section and spares much additional work. Also, it is possible to obtain much better estimates for a general $K \in \boldsymbol{F}^M(\mathfrak{N})$ than those of Corollary 8.19, but for the same reason we do not want to go into details here. We hope to return to these questions in the next paper.

9. L^p-Multipliers

We turn to the main multiplier theorem. This part is strongly influenced by Duoandikoetxea-Rubio de Francia [4].

Theorem 9.1. *Let $K \in \mathcal{S}'(\mathfrak{g})$ be such that $\widehat{K}$ is locally integrable, smooth away from $H^\star$, and satisfies the estimates*

$$|D^\alpha \widehat{K}(\xi)| \le C_\alpha \prod_{k=1}^{N} N_k(\xi)^{-p_k\alpha_k}, \qquad \xi \notin H^\star, \tag{9.2}$$

where $N_k(\xi) = \sum_{j\succeq k} |\xi_j|^{1/p_j}$ and

$$H^\star = \{\xi \in \mathfrak{g}^\star : \exists_{k\in\mathcal{N}_{\max}}\ \xi_k = 0\}.$$

Then, for $1 < p < \infty$,

$$\|f \star K\|_{L^p(\mathfrak{g})} \le C_K \|f\|_{L^p(\mathfrak{g})}, \qquad f \in C_c^\infty(\mathfrak{g}),$$

where C_K depends on a finite number of constants C_α. Thus, $f \mapsto f \star K$ extends to a bounded operator on $L^p(\mathfrak{g})$.

Having established the calculus of $\mathfrak{N}$-kernels, we just may invoke the argument from [10]. However, for the convenience of the reader, we give a sketch of the proof.

Let $\mathcal{N}_k = \{j \in \mathcal{N} : j \succeq k\}$, for $k \in \mathcal{N}$, and $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$. Let $\mathfrak{g}_k$ be the linear subspace of $\mathfrak{g}$ generated by the vectors e_j, where $j \in \mathcal{N}_k$ (see Lemma 5.8). By Dziubański [5], [6] and Lemma 7.2 of [10], for every $k \in \mathcal{N}$, there exists an even real Schwartz function φ_k on $\mathfrak{g}_k$ such that

$$\int_{\mathfrak{g}_k} x^\alpha \varphi_k(x)dx = 0, \qquad \alpha \in \boldsymbol{N}^{|\mathcal{N}_k|},$$

and

$$\int_0^\infty (\varphi_k)_t \star (\varphi_k)_t \star f \,\frac{dt}{t} = f$$

in $L^2(\mathfrak{g}_k)$. Recall that $\varphi_t(x) = t^{-Q}\varphi(\delta_{t^{-1}}x)$. Let $\Phi_k = \varphi_k \otimes \delta_k$, where δ_k is the Dirac delta at 0 on $\mathfrak{g}_k^\perp = \langle e_j\rangle_{j\notin\mathcal{N}_k}$.

Lemma 9.3. *For every admissible $M \in \boldsymbol{R}^N$,*

$$\Phi = \Phi_1 \star \Phi_2 \star \cdots \star \Phi_N \in \mathcal{S}(\mathfrak{g}) \cap \boldsymbol{F}^M(\mathfrak{N}).$$

Proof. Let $\mathbf{g}$ be the metric on $\mathfrak{g}^\star$ determined by the filtration $\mathfrak{N}$. Then, for every $k \in \mathcal{N}$ and every $r > 0$, $\widehat{\Phi_k} \in S(g_k^{-r}, \mathbf{g})$. By Corollary 7.8, $\widehat{\Phi} \in S(\boldsymbol{m}^{-r}, \mathbf{g})$, where

$$\boldsymbol{m}(\xi) = g_1(\xi)g_2(\xi)\dots g_N(\xi) \ge 1 + \rho(\xi), \qquad \xi \in \mathfrak{g}^\star,$$

which implies $\widehat{\Phi} \in \mathcal{S}(\mathfrak{g}^\star)$. Hence, $\Phi \in \mathcal{S}(\mathfrak{g})$.

If $m_k \le 0$, for $k \in \mathcal{N}$, then $\Phi \in \mathcal{S}(\mathfrak{g})$ implies $\Phi \in \boldsymbol{F}^M(\mathfrak{N})$. Otherwise, we need vanishing moments of φ_k. Let $\{a_k\}_{k\in\mathcal{N}}$ be an admissible set of multiindices such that $(a_k)_j = 0$ for $j \notin \mathcal{N}_k$ and $\sigma(a_k) = m_k$. If φ_k has vanishing moments, then $\Phi_k \in \boldsymbol{F}^{a_k}(\mathfrak{N})$, so, by Theorem 8.15, $\Phi \in \boldsymbol{F}^M(\mathfrak{N})$. □

The following proposition collects together the results described and proved in [10], Sections 7 and 8. Even though the kernels considered in [10] are less general, the proofs stay valid with just cosmetic changes.

Proposition 9.4. *For $T = (t_1, t_2, \dots, t_N) \in \boldsymbol{R}_+^N$, let*

$$\Phi_T(x) = (\Phi_1)_{t_1} \star (\Phi_2)_{t_2} \star \dots (\Phi_N)_{t_N}(x), \qquad x \in \mathfrak{g}.$$

1) *For every admissible $M \in \boldsymbol{R}^N$, $\Phi_T \in \boldsymbol{F}^M(\mathfrak{g})$, and*

$$\|\Phi_T\|_{\boldsymbol{F}^M, l} \lesssim \prod_{k=1}^N t_k^{m_k}, \qquad l \in \boldsymbol{N},$$

uniformly in $T \in \boldsymbol{R}_+^N$;

2) *For $f, g \in L^2(\mathfrak{g})$,*

$$\langle f, g \rangle = \int_{\boldsymbol{R}_+^N} \langle f \star \Phi_T, g \star \Phi_T \rangle \frac{dT}{|T|},$$

where

$$\frac{dT}{|T|} = \frac{dt_1 dt_2, \dots, dt_N}{t_1 t_2 \dots t_N};$$

3) *The Littlewood-Paley operator*

$$G_\Phi(f)(x) = \left(\int_{\boldsymbol{R}_+^N} |f \star \Phi_T(x)|^2 \frac{dT}{|T|} \right)^{1/2}$$

is of type (p, p), for every $1 < p < \infty$.

For $T, S \in \boldsymbol{R}_+^N$, let $TS = (t_1 s_1, t_2 s_2, \dots, t_N s_N)$. Let

$$K_{T,S} = \widetilde{\Phi}_{TS} \star K \star \Phi_T, \qquad T, S \in \boldsymbol{R}_+^N,$$

and

$$\mathcal{K}_S^\star f(x) = \sup_T |f \star K_{T,S}(x)|, \qquad S \in \boldsymbol{R}_+^N.$$

Note that, by Proposition 8.14 and Lemma 9.3, $K_{T,S}$ is a continuous and square-integrable function. Let us also define

$$\gamma(T) = \prod_{k=1}^N \gamma(t_k), \qquad T = (t_1, t_2, \dots, t_N) \in \boldsymbol{R}_+^N,$$

where $\gamma(t) = \min\{t, t^{-1}\}$. Let

$$F(x) = \prod_{k=1}^N \gamma(|x_k|)^{1/4} |x_k|^{-1}$$

and

$$F_T(x) = \left(\prod_{k \in \mathcal{N}} t_k^{-p_k} \right) F(\delta_{t_1^{-1}} x_1, \delta_{t_2^{-1}} x_2, \dots, \delta_{t_N^{-1}} x_N),$$

for $T = (t_1, t_2, \dots, t_N)$.

Theorem 9.5 (Christ). *The maximal operator*

$$\mathcal{M}_F f(x) = \sup_T |f \star F_T(x)|.$$

is of strong type (p, p)*, for* $1 < p < \infty$*.*

For the proof, see Christ [2].

Proposition 9.6. *Let* $1 < p < \infty$*. Then, there exists* $l > 0$ *such that, for every* S *and every* T *in* $\boldsymbol{R}_+^N$*,*

$$|K_{T,S}(x)| \lesssim \gamma(S)^{1/4} F_T(x), \qquad x \in \mathfrak{g},$$

and

$$\|\mathcal{K}_S^\star f\|_{L^p} \lesssim \gamma(S)^{1/4} \|f\|_{L^p}, \qquad f \in L^p(\mathfrak{g}).$$

Proof. The second part follows from (9.5) and the first part. The first part is a consequence of 1) of Proposition 9.4, Theorem 8.15, and Corollary 8.19. □

Propositions (9.4) and (9.6) give everything that is needed to complete the proof of Theorem 9.1. Here is the conclusion of the proof with some shortcuts. There is no loss of generality in assuming that $\widetilde{K} = K$.

Let $f, g \in \mathcal{S}(\mathfrak{g})$. By 2) of Proposition 9.4,

$$\langle f \star K, g \rangle = \int_{\boldsymbol{R}_+^d} \int_{\boldsymbol{R}_+^d} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T} \frac{dS}{S},$$

where $f_T = f \star \Phi_T$. We want to estimate

$$L_S(f, g) = \int_{\boldsymbol{R}_+^d} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T},$$

for a given S. Recall that, by 3) of Proposition 9.4, the square function operator G_Φ is of type (p, p), for every $1 < p < \infty$. Let $1 < p < 2$. By the Schwartz and Hölder inequalities,

$$|L_S(f, g)| \lesssim \|f\|_p \left\| \int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2}^{1/2},$$

where $1/p + 1/q = 1$. Note that $q > 2$. Thus, there exists a nonnegative function u with $\|u\|_r = 1$, where $2/q + 1/r = 1$, such that

$$\begin{aligned} A &= \left\| \int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2} \\ &= \int_{\mathfrak{g}} \int_{\boldsymbol{R}_+^N} |g_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} u(x)\, dx \\ &\leq \int_{\mathfrak{g}} \int_{\boldsymbol{R}_+^N} \|K_{T,S}\|_1 |g_{TS}|^2 \star |K_{T,S}|(x)| \frac{dT}{T} u(x)\, dx. \end{aligned}$$

Therefore, by Proposition 9.6, there exists $l \in \boldsymbol{N}$ such that

$$\begin{aligned} A &\lesssim \|K\|_{\boldsymbol{F}^0,l}\gamma(S)^{1/4}\int_{\mathfrak{g}}\int_{\boldsymbol{R}_+^d}|g_{TS}(x)|^2\frac{dT}{T}K_S^\star u(x)\,dx \\ &\lesssim \|K\|_{\boldsymbol{F}^0,l}\gamma(S)^{1/4}\|G_\Phi(g)\|_q^2\|K_S^\star u\|_r \\ &\lesssim \|K\|_{\boldsymbol{F}^0,l}^2\gamma(S)^{1/2}\|g\|_q^2, \end{aligned}$$

whence, by 3) of Proposition 9.4,

$$|L_S(f,g)| \lesssim \|K\|_{\boldsymbol{F}^0,l}\gamma(S)^{1/4}\|f\|_p\|g\|_q. \tag{9.7}$$

Finally,

$$\begin{aligned} |\langle f\star K, g\rangle| &\lesssim \|K\|_{\boldsymbol{F}^0,l}\left(\int_{\boldsymbol{R}_+^d}\gamma(S)^{1/4}\frac{dS}{S}\right)\|f\|_p\|g\|_q \\ &\approx \|K\|_{\boldsymbol{F}^0,l}\|f\|_p\|g\|_q, \end{aligned}$$

which proves our case for $1 < p < 2$. The result for $2 < p < \infty$ follows by duality. The case $p = 2$ has already been established in Theorem 7.12.

10. Appendix. Convolution of distributions.

Let X be an N-dimensional vector space as described in Section 2.

10.1 (Sobolev inequality). *We have*

$$\|f\|_{A(X)} \lesssim \max_{|\alpha|\le N/2+1}\|D^\alpha f\|_{L^2(X)}, \qquad f \in \mathcal{S}(X),$$

where $\|f\|_{A(X)} = \int_{\mathfrak{g}^\star}|\widehat{f}(\xi)|d\xi$. (Proposition 3.5.14 of Narasimhan [17].)

The following is a direct consequence of (10.1).

10.2. *Let F be a measurable function on an open subset of X. If, for every $\alpha \in \mathcal{A}_{\boldsymbol{N}}$, $D^\alpha F$ is a locally bounded function, then F is smooth.*

Let $\mathfrak{g}$ be a nilpotent Lie group as described in Section 5. The following definition of the general convolution is due to C. Chevalley. See Chevalley [1], Section 8.

Definition 10.3. We say that distributions $S, T \in \mathcal{S}'(\mathfrak{g})$ are convolvable if

$$\int_{\mathfrak{g}}\left|\left(\widetilde{S}\star f\right)(x)\left(T\star\widetilde{g}\right)(x)\right|dx < \infty, \qquad f, g \in \mathcal{S}(\mathfrak{g}).$$

Proposition 10.4. *If S, T are convolvable, then there exists a unique distribution $S\star T$ such that*

$$\langle S\star T, f\star g\rangle = \int_{\mathfrak{g}}\left(\widetilde{S}\star f\right)(x)\left(T\star\widetilde{g}\right)(x)dx. \qquad f, g \in \mathcal{S}(\mathfrak{g}). \tag{10.5}$$

10.6. *If $S, T \in \mathcal{S}'(\mathfrak{g})$ are convolvable, then*

$$(S\star T)\star\varphi = S\star(T\star\varphi), \qquad \varphi\star(S\star T) = (\varphi\star S)\star T, \tag{10.7}$$

for $\varphi \in \mathcal{S}(\mathfrak{g})$. Moreover, $\widetilde{T}$ and $\widetilde{S}$ are also convolvable and

$$(S\star T)^\sim = \widetilde{T}\star\widetilde{S}.$$

10.8. *A distribution $R \in \mathcal{S}'(\mathfrak{g})$ is said to be an $\mathcal{S}$-convolver if $R \star f, f \star R \in \mathcal{S}(\mathfrak{g})$, for every $f \in \mathcal{S}(\mathfrak{g})$.*

10.9. *If $K \in \mathcal{S}'(\mathfrak{g})$ and R is an $\mathcal{S}$-convolver, then K, R and R, K are convolvable and*

$$\langle K \star R, f\rangle = \langle K, f \star \widetilde{R}\rangle, \qquad \langle R \star K, f\rangle = \langle K, \widetilde{R} \star f\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

10.10. *A distribution $R \in \mathcal{S}'(\mathfrak{g})$ is central if $R \star f = f \star R$, for every $f \in \mathcal{S}(\mathfrak{g})$. If R is a central $\mathcal{S}$-convolver, then $R \star K = K \star R$, for every $K \in \mathcal{S}'(\mathfrak{g})$.*

10.11. *If S and T are convolvable and R is an $\mathcal{S}$-convolver, then S, $T \star R$ and $R \star S$, T are convolvable, and*

$$(S \star T) \star R = S \star (T \star R), \qquad R \star (S \star T) = (R \star S) \star T.$$

If, moreover, R is central, then

$$(S \star R) \star T = S \star (R \star T).$$

Acknowledgements

I wish to thank L. Newelski, M. Pascu, and E. M. Stein for their advice on the subject of the paper.

References

[1] C. Chevalley, *Theory of distributions*, Columbia University, 1950/51,

[2] M. Christ, *The strong maximal function on a nilpotent group*, Trans. Am. Math. Soc., **331** (1992), 1–13,

[3] L. Corwin, F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Basic theorey and examples*, Cambridge University Press, Cambridge 1990,

[4] J. Duoandikoetxea and Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Inv. math., **84** (1986), 541–561,

[5] J. Dziubański, *Schwartz spaces associted with some non-differential convolution operators on homogeneous groups*, Colloq. Math., 63 (1992), 153-157,

[6] J. Dziubański, *A note on Schrödinger operators with polynomial potentials*, Colloq. Math., 78 (1998), 149-161,

[7] G.B. Folland and E.M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton 1982,

[8] P. Głowacki, *The Melin calculus for general homogeneous groups*, Arkiv för matematik, **45** (2007), 31-48,

[9] P. Głowacki, *Composition and L^2-boundedness of flag kernels*, Colloq. Math., **118** (2010), 581–585, *Correction*, Colloq. Math., **120** (2010), 331,

[10] P. Głowacki *L^p-boundedness of flag kernels on homogeneous groups via symbolic calculus*, J. Lie Theory, **23** (2013), 953-977,

[11] G. A. Hunt, *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264-293,

[12] L. Hörmander, *The Weyl calculus of pseudodifferential operators*, Comm. Pure Appl. Math. 32 (1979), 359-443;

[13] D. Manchon, *Formule de Weyl pour les groupes de Lie nilpotents*, J. Reine Angew. Math. 418 (1991), 77-129;

[14] A. Melin, *Parametrix constructions for right-invariant differential operators on nilpotent Lie groups*, Ann. Glob. Anal. Geom. 1 (1983), 79-130;

[15] A. Nagel, F. Ricci, and E.M. Stein, *Singular integrals with flag kernels and analysis on quadratic CR manifolds*, J. Func. Analysis,**181**, (2001), 29–18,

[16] A. Nagel, F. Ricci, E.M. Stein, S. Wainger, *Singular integrals with flag kernels on homogeneous groups I*, Revista Mat. Iberoam. 28 (2012), 631-722,

[17] R. Narasimhan, *Analysis on real and complex manifolds*, Masson & CIE, Editeur – Paris, North-Holland Publishing Company, Amsterdam 1968.
[18] L. Schwartz, *Théorie des distributions*, Paris, Hermann, 1966,
[19] E.M. Stein, *Harmonic Analysis*, Princeton University Press, Princeton NJ, 1993;
[20] K. Yosida, *Functional analysis*, Springer-Verlag, Berlin-Heidelberg-New York 1980.

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-386 Wrocław, Poland
Email address: `glowacki@math.uni.wroc.pl`