In memory of Tadek Pytlik, our teacher and friend

Sobolev spaces related to Schrödinger operators with polynomial potentials

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Abstract

The aim of this note is to prove the following theorem.

Let

$$Af(x) = P(D)f(x) + V(x)f(x),$$

where $P(ix)$ is a nonnegative homogeneous elliptic polynomial on $\mathbf{R}^d$ and V is a nonnegative polynomial potential. Then for every $1 < p < \infty$ and every $\alpha > 0$ there exist constants $C_1, C_2 > 0$ such that

$$\|P(D)^\alpha f\|_{L^p} + \|V^\alpha f\|_{L^p} \leq C_1 \|A^\alpha f\|_{L^p}$$

and

$$\|A^\alpha f\|_{L^p} \leq C_2 \| \left(P(D)^\alpha + V^\alpha\right) f\|_{L^p}$$

for $f \in C_c^\infty(\mathbf{R}^d)$.

We take advantage of the Christ inversion theorem for singular integral operators with a small amount of smoothness on nilpotent Lie groups, the maximal subelliptic L^2-estimates for the generators of stable semi-groups of measures, and the principle of transference of Coifman-Weiss.

Key words: Schrödinger operators, nilpotent Lie groups, Sobolev spaces, singular integrals, stable semigroups of measures, maximal subelliptic estimates

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1 Introduction

Let

$$\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

be the (positive) Laplace operator on $\mathbf{R}^d$. Let

$$Lf(x) = \Delta f(x) + V(x)f(x)$$

be a Schrödinger operator with a nonnegative polynomial potential V. The following maximal L^p-estimates

$$\|\Delta^\alpha f\|_{L^p} + \|V^\alpha f\|_{L^p} \le C_1 \|L^\alpha f\|_{L^p} \tag{1}$$

and

$$\|L^\alpha f\| \le C_2 \| \left(\Delta^\alpha + V^\alpha\right) f\|_{L^p} \tag{2}$$

for $1 < p < \infty$ and $\alpha > 0$, attracted attention of a number of authors. Let us briefly recall some results. The estimate (1) is due to Nourrigat [16] ($p = 2$ and $\alpha = 1$), Guibourg [14] ($1 < p < \infty$, $\alpha = 1$) and Zhong [18] ($1 < p < \infty$, $\alpha > 0$). In the case of the Hermite operator estimates similar to (2) were recently obtained for positive integers α by Bongioanni and Torrea [2] by using the Mehler kernel.

Under much less restrictive assumption that V belongs to the reverse Hölder class B_q, Shen [17] obtained the estimates (1) for $\alpha = 1, 1/2$, $q > d/2$, $d \ge 3$, and the range of p depending on q. These were subsequently generalized by Auscher and Ben Ali [1] for $0 < \alpha \le 1$, $q > 1$, and $1 < p < 2(q+\varepsilon)$. They also succeeded in obtaining the estimates

$$\|L^\alpha f\| \le C_3 \left(\|\Delta^\alpha f\|_{L^p} + \|V^\alpha f\|_{L^p}\right), \qquad 1 < p < \infty, \quad \alpha = 1/2.$$

Note that if V is a polynomial, then $V \in \bigcap_{q>1} B_q$, and the estimates (1) of Auscher-Ben Ali hold for all $1 < p < \infty$.

Nourrigat uses the method of representations of nilpotent Lie groups, whereas Guibourg works with the machinery of Hörmander's slowly varying metrics. The methods applied by Shen and Auscher-Ben Ali include the Fefferman-Phong inequalities, the Calderón-Zygmund decompositions, and various techniques of interpolation.

Let

$$\delta_t x = (t^{m_1} x_1, t^{m_2} x_2, ..., t^{m_d} x_d), \qquad x = (x_1, x_2, ..., x_d),$$

be a family of dilations on $\mathbf{R}^d$, where $m_1, m_2, ..., m_d$ are positive integers. Let $P(x)$ be a homogeneous polynomial such that

$$P(ix) > 0 \quad \text{for } x \ne 0. \tag{3}$$

We consider the Schrödinger type operator

$$A = P(D) + V(x)$$

on $\mathbf{R}^d$ with a nonnegative polynomial potential $V(x)$.

The aim of this note is to prove the following theorem.

Theorem 1 *For every $1 < p < \infty$ and every $\alpha > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$\|P(D)^\alpha f\|_{L^p} + \|V^\alpha f\|_{L^p} \le C_1 \|A^\alpha f\|_{L^p} \tag{4}$$

and

$$\|A^\alpha f\|_{L^p} \le C_2 \| \left(P(D)^\alpha + V^\alpha\right) f\|_{L^p} \tag{5}$$

for f in the Schwartz class $\mathcal{S}(\mathbf{R}^d)$.

The estimate (5) with the full range of $\alpha > 0$ seems to be new even for $P(D) = \Delta$. A typical example of a constant coefficient homogeneous differential operator occurring in the theorem is

$$P(D) = \sum_{j=1}^{d} (-1)^{n_j} \frac{\partial^{2n_j}}{\partial x_j^{2n_j}}, \qquad P(ix) = \sum_{j=1}^{d} x_j^{2n_j},$$

The essential part of our considerations is limited to the analysis on homogeneous nilpotent groups. We take advantage of the Christ inversion theorem for singular integral operators with a small amount of smoothness, the maximal subelliptic L^2-estimates for the generators of stable semi-groups of measures, and the principle of transference of Coifman-Weiss. We believe that the combination of the above-mentioned means applied in this context may be interesting.

2 Preliminaries

Let $\mathcal{N}$ be a finite-dimensional homogeneous group endowed with a family of dilations $\{\delta_t\}_{t>0}$ and a homogeneous norm $x \mapsto |x|$ which is smooth away from the identity. Let dx denote Haar measure on $\mathcal{N}$ and D the homogeneous dimension of $\mathcal{N}$. Thus $d(\delta_t x) = t^D dx$. Let

$$\Sigma = \{x \in \mathcal{N} : |x| = 1\}$$

be the unit sphere relative to the homogeneous norm. For a nonzero $x \in \mathcal{N}$, let

$$\bar{x} = \delta_{|x|^{-1}} x.$$

There exists a unique Radon measure $d\bar{x}$ on Σ such that for all continuous functions f on $\mathcal{N}$ with compact support

$$\int_{\mathcal{N}} f(x)dx = \int_0^\infty r^{D-1} \int_\Sigma f(\delta_r \bar{x}) d\bar{x} dr.$$

Since $\mathcal{N}$ is connected and simply connected nilpotent Lie group it may be identified *via* the exponential map with its Lie algebra. We shall stick to this identification throughout the paper and think of $\mathcal{N}$ as being a nilpotent Lie algebra with the Campbell-Hausdorff multiplication. Let us remark that our convention implies that the origin 0 plays the rôle of the group identity and $-x$ is the inverse of $x \in \mathcal{N}$. Moreover, the dilations δ_t are also automorphisms of the Lie algebra structure of $\mathcal{N}$. For more on homogeneous groups the reader is referred to Folland-Stein [9].

Let $\mathbf{V}$ be a homogeneous (that is invariant under dilations) ideal in the Lie algebra $\mathcal{N}$ and $\mathbf{S}$ its homogeneous linear complement. There exist homogeneous polynomial mappings

$$\nu : \mathcal{N} \to \mathbf{V}, \qquad \sigma : \mathcal{N} \to \mathbf{S}$$

such that every element $a \in \mathcal{N}$ decomposes uniquely as $a = \nu(a)\sigma(a)$, while the multiplication

$$x \circ y = \sigma(xy), \qquad x, y \in \widetilde{\mathcal{N}},$$

makes $\mathbf{S}$ into a homogeneous group isomorphic to $\mathcal{N}/\mathbf{V}$ with σ being the canonical homomorphism. Dilations on $\mathbf{S}$ are simply those of $\mathcal{N}$ restricted to $\mathbf{S}$.

Let ω be a linear functional on the vector space $\mathbf{V}$. Then the representation $\pi^{(\mathbf{V},\omega)} = \pi^\omega$ of $\mathcal{N}$ induced by the character $e^{i\langle \omega, v\rangle}$ on $\mathbf{V}$ is defined by

$$\pi_a^\omega f(x) = e^{i\langle \nu(xa), \omega\rangle} f(\sigma(xa)), \tag{6}$$

where $f \in \mathcal{S}(\mathbf{S})$. It goes without saying that π^ω can be understood as a uniformly bounded representation of $\mathcal{N}$ on the Banach space $L^p(\mathbf{S})$ for every $1 \le p < \infty$. If $p = 2$, then π^ω is unitary.

A tempered distribution T on $\mathcal{N}$ is said to be a *kernel of order* $r \in \mathbf{R}$ if it coincides with a Radon measure away from the origin and satisfies

$$\langle T, f \circ \delta_t \rangle = t^r \langle T, f \rangle$$

for $f \in C_c^\infty(\mathcal{N})$ and $t > 0$. Note that, by homogeneity, any kernel of order $r > 0$ coincides with a bounded measure outside any neighbourhood of the origin and thus extends to a continuous linear form on the space $C_b^\infty(\mathcal{N})$ of bounded smooth functions on $\mathcal{N}$ with natural topology. A kernel T of order

$r \in \mathbf{R}$ is called *regular* if it coincides with a C^∞ function away from the origin. Any kernel T of order r gives a rise to a convolution operator $f \mapsto f \star T$ which will be denoted by the same symbol T. If T is regular and symmetric, then the operator T is essentially selfadjoint on $L^2(\mathcal{N})$ with $\mathcal{S}(\mathcal{N})$ for its core domain. If T is a kernel of order $r > 0$, then the operator

$$\pi_T = \pi_T^\omega : \mathcal{S}(\mathbf{S}) \to L^2(\mathbf{S})$$

is defined by

$$\langle \pi_T f, g \rangle = \langle T, \varphi_{f,g} \rangle, \qquad\qquad f, g \in \mathcal{S}(\mathbf{S}),$$

where $\varphi_{f,g}(x) = \langle \pi_x f, g \rangle$ is in $C_b^\infty(\mathcal{N})$. The operator π_T is closable.

A real distribution T on $\mathcal{N}$ is said to be *accretive* if

$$\langle T, f \rangle \geq 0$$

for all real $f \in C_c^\infty(\mathcal{N})$ that take on their maximal value at the identity. It follows directly from the definition that such a T coincides with a negative Radon measure away from the origin which is bounded on the complement of any neighbourhood of the origin. Thus every accretive T extends by continuity to a linear form on $C_b^\infty(\mathcal{N})$.

A distribution T is accretive if and only if there exists a unique continuous semigroup of subprobability measures $\{\mu_t\}_{t>0}$ for which $-T$ is the *generating functional*, that is,

$$\langle T, f \rangle = -\frac{d}{dt}\bigg|_{t=0} \langle \mu_t, f \rangle$$

for $f \in C_c^\infty(\mathcal{N})$. If, in addition, T is a kernel of order $r > 0$, then μ_t are probability measures.

Recall that if T is a symmetric accretive kernel of order $r > 0$ with the semigroup $\{\mu_t\}$, then for every $0 < a < 1$ the formula

$$\langle T^a, f \rangle = \frac{1}{\Gamma(-a)} \int_0^\infty t^{-1-a} \langle \delta_0 - \mu_t, f \rangle \, dt$$

where δ_0 stands for the Dirac delta at the origin, defines an accretive kernel T^a of order ar. An arbitrary positive power T^ℓ of a regular accretive distribution T is defined by $T^\ell = T^n \star T^{\ell - n}$, where n is the integer part of $\ell > 0$.

3 Singular Integrals

For $q \in (1, \infty)$ let A_q denote the set of all operators $T : \mathcal{S}(\mathcal{N}) \to \mathcal{S}'(\mathcal{N})$ of the form $Tf = cf + f \star K$, where K is a principal value distribution that coincides with a locally L^q-function away from the origin, that is,

$$\langle K, f\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\Omega(x)}{|x|^D} f(x)\,dx, \tag{7}$$

where Ω is a homogeneous function of degree 0, $\int_\Sigma \Omega(\bar{x})d\bar{x} = 0$, $\Omega \in L^q(\Sigma)$.

Let $\eta \in C_c^\infty(\mathcal{N})$ be such that $\eta(x) = \eta(|x|)$, $\operatorname{supp}\eta \subset \{x : 2^{-1} < |x| < 2\}$, $\sum_{j\in\mathbf{Z}} \eta(\delta_{2^j} x) = 1$ for $x \neq 0$. Set $k_j(x) = \eta(\delta_{2^j}x)\Omega(x)|x|^{-D}$. For integers $N \le M$ let $K_{N,M}(x) = \sum_{j=N}^M k_j(x)$. Obviously,

$$\langle K, f\rangle = \lim_{N\to-\infty, M\to\infty} \int K_{N,M}(x) f(x)\,dx, \qquad f \in \mathcal{S}(\mathcal{N}).$$

Much of our argument relies on the following results of M. Christ.

Theorem 2 *Let $1 < q < \infty$. If $T \in A_q$, then $f \mapsto f \star T$ extends to a bounded operator $T : L^p(\mathcal{N}) \to L^p(\mathcal{N})$ for every $1 < p < \infty$. Moreover, A_q is a Banach algebra with the norm $|T|_q = |c| + \|\Omega\|_{L^q(\Sigma)}$.*

This is a compilation of Propositions 1 and 2 of Christ [3].

Theorem 3 *Let $1 < q < \infty$. If $T \in A_q$ is invertible on $L^2(\mathcal{N})$, then its inverse T^{-1} belongs to A_q as well.*

This is Theorem 3 of Christ [3].

Theorem 4 *Let $1 < q < \infty$ and K be of the form (7). For every $p \in (1,\infty)$ there exists a constant C_p such that for every integers $N \le M$,*

$$\|f \star K_{N,M}\|_{L^p(\mathcal{N})} \le C_p \|f\|_{L^p(\mathcal{N})}. \tag{8}$$

Moreover, the kernels k_j satisfy the assumptions of the Cotlar-Stein lemma, that is,

$$\|f \star k_j \star k_i^*\|_{L^2(\mathcal{N})} + \|f \star k_j^* \star k_i\|_{L^2(\mathcal{N})} \le C 2^{-\varepsilon|j-i|} \|f\|_{L^2(\mathcal{N})}. \tag{9}$$

This is a compilation of Lemma 2.10 of Christ [3] and Lemma 7 of Christ [4].

The purpose of the following considerations is to justify apparently obvious definitions concerning representations of some unbounded convolution operators and their seemingly obvious algebra. The reader not interested in technical details may skip the remaining part of this section provided he accepts Propositions 8 and 9.

¿From now on we assume that K is a principal value distribution of the form (7).

Proposition 5 *For every $\varphi \in \mathcal{S}(\mathcal{N})$ there exists a homogeneous of degree 0 function $\widetilde{\Omega} \in L^q(\Sigma)$, such that*

$$\sup_{N,M} |K_{N,M} \star \varphi(x)| \leq \frac{\widetilde{\Omega}(x)}{(1+|x|)^D}. \tag{10}$$

PROOF.

By the mean value theorem (cf. Folland-Stein [9], page 28), for every $m > 0$ there exists a constant C_m (that depends also on φ) such that

$$|\varphi(yx) - \varphi(x)| \leq C_m |y|(1+|x|)^{-m} \quad \text{for } |y| \leq 2.$$

Hence for $j \geq 0$ one has

$$\begin{aligned} |k_j \star \varphi(x)| &= \left| \int k_j(y^{-1})(\varphi(yx) - \varphi(x))\, dy \right| \\ &\leq C'_m 2^{-j}(1+|x|)^{-m}. \end{aligned} \tag{11}$$

Set $F(x) = \sum_{j<0} |k_j(x)|$. Then $F(x) \leq C|\Omega(x)|(1+|x|)^{-D}$, and, consequently, $F \in L^q(\mathcal{N})$. Thus $F \star |\varphi|$ is a bounded continuous function. For $m > 2D$,

$$\begin{aligned} F \star |\varphi|(x) &\leq C \int_{\mathcal{N}} \frac{|\Omega(xy^{-1})|}{(1+|xy^{-1}|)^D}(1+|y|)^{-m}\, dy \\ &\leq \int_{|y|\leq|x|/C} + \int_{|x|/C\leq|y|\leq C|x|} + \int_{|y|>C|x|} \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{12}$$

where

$$I_1 + I_3 \leq C(1+|x|)^{-D} \int_{\mathcal{N}} \frac{|\Omega(xy^{-1})|}{(1+|y|)^m}\, dy \leq C(1+|x|)^{-D}\mathcal{M}\Omega(x), \tag{13}$$

with $\mathcal{M}$ standing for the Hardy-Littlewood maximal function, and

$$I_2 \leq C(1+|x|)^{-m/2} \int_{\mathcal{N}} \frac{|\Omega(xy^{-1})|}{(1+|y|)^{m/2}}\, dy \leq C(1+|x|)^{-D}\mathcal{M}\Omega(x). \tag{14}$$

Since $\mathcal{M}\Omega$ is homogeneous of degree 0 function and belongs to $L^q_{loc}(\mathcal{N})$, we get the assertion.

Recall that D is the homogeneous dimension of $\mathcal{N}$. Let D_1 and D_2 be those of $\mathbf{V}$ and $\mathbf{S}$, respectively. Of course, $D = D_1 + D_2$.

Lemma 6 *Let $1/p+1/p'=1$, where $p,p'>1$. Let k be a measurable function on $\mathcal{N}$ such that*

$$|k(a)| \leq \frac{\Omega(a)}{(1+|a|)^{D-\delta}}, \tag{15}$$

where $\Omega \in L^p(\Sigma)$ is homogeneous of degree 0, and $0<\delta<D_2/p'$. Then, for every $\phi \in L^{p'}(\mathbf{S})$,

$$\int_{\mathcal{N}} |k(a)\phi(\sigma(a))| da \leq C\|\phi\|_{L^{p'}(\mathbf{S})}.$$

PROOF. Let us pick ε such that

$$\frac{(p-1)D_1}{D-\delta} < \varepsilon < p - \frac{D}{D-\delta}. \tag{16}$$

Let $v=\nu(a)$, $s=\sigma(a)$. Then

$$I = \int_{\mathcal{N}} |k(a)\phi(\sigma(a))| da \leq C_0 \iint_{\mathbf{V}\times\mathbf{S}} \frac{\Omega(a)|\phi(s)|\, ds\, dv}{(1+|v|+|s|)^{D-\delta}},$$

where $|v|$, $|s|$ are homogeneous norms in $\mathbf{V}$ and $\mathbf{S}$ respectively. Therefore, by a double application of the Hölder inequality,

$$\begin{aligned}
I &\leq C_0\|\phi\|_{L^{p'}(\mathbf{S})} \int_{\mathbf{V}} \Big(\int_{\mathbf{S}} \frac{\Omega(v,s)^p\, ds}{(1+|s|+|v|)^{p(D-\delta)}} \Big)^{1/p} dv \\
&\leq C_0\|\phi\|_{L^{p'}(\mathbf{S})} \int_{\mathbf{V}} \frac{1}{(1+|v|)^{\varepsilon(D-\delta)/p}} \Big(\int_{\mathbf{S}} \frac{\Omega(v,s)^p\, ds}{(1+|s|+|v|)^{(p-\varepsilon)(D-\delta)}} \Big)^{1/p} dv \\
&\leq C_1\|\phi\|_{L^{p'}(\mathbf{S})} \Big(\int_{\mathbf{V}} \frac{dv}{(1+|v|)^{\varepsilon p'(D-\delta)/p}} \Big)^{1/p'} \Big(\iint_{\mathbf{V}\times\mathbf{S}} \frac{\Omega(v,s)^p\, ds\, dv}{(1+|v|+|s|)^{(p-\varepsilon)(D-\delta)}} \Big)^{1/p} \\
&\leq C\|\phi\|_{L^{p'}(\mathbf{S})}
\end{aligned}$$

since, by (16),

$$\varepsilon p'(D-\delta)/p > D_1, \qquad (p-\varepsilon)(D-\delta) > D.$$

Proposition 7 *Let $p>1$. Let k, l be measurable functions on $\mathcal{N}$ such that*

$$|k(a)| \leq \frac{\Omega_1(a)}{(1+|a|)^D}, \qquad |l(a)| \leq \frac{\Omega_2(a)}{(1+|a|)^D}, \tag{17}$$

where $\Omega_j \in L^p(\Sigma)$ are homogeneous of degree 0. Let $\phi \geq 0$ be a measurable function on $\mathbf{S}$ such that

$$\int_{\mathbf{S}} \phi(s)^{p'} (1+|s|)^{\varepsilon p'}\, ds < \infty$$

for some $\varepsilon > D_2/2p'$. Then

$$\int_{\mathcal{N}} \int_{\mathcal{N}} |k(x)l(y)\phi(\sigma(xy))|\, dx\, dy < \infty.$$

PROOF. By diminishing ε, if necessary, we may assume that

$$\frac{D_2}{2p'} < \varepsilon < \frac{D_2}{p'}. \tag{18}$$

By using the inequality

$$1 \le C(1+|\sigma(x)|)^{-1}(1+|\sigma(y)|)(1+|\sigma(xy)|),$$

we get

$$|k(x)l(y)\phi(\sigma(xy))| \le C_1|k_1(x)l_1(y)\phi_1(\sigma(xy))|,$$

where

$$k_1(x) = (1+|\sigma(x)|)^{-\varepsilon}k(x), \quad l_1(y) = (1+|\sigma(y)|)^{\varepsilon}l(y), \quad \phi_1(s) = (1+|s|)^{\varepsilon}\phi(s).$$

Thus, by (18), l_1 satisfies (15), while

$$\Big(\int_{\mathbf{S}} \phi_1(\sigma(xs))^{p'}\,ds\Big)^{1/p'} = \|\phi_1\|_{L^{p'}(\mathbf{S})} < \infty, \quad x \in \mathcal{N}.$$

Recall that $\sigma(x\sigma(y)) = \sigma(xy)$. Applying Lemma 6 we get

$$\int_{\mathcal{N}}\int_{\mathcal{N}} |k(x)l(y)\phi(\sigma(xy))|\,dy\,dx \le C_2\|\phi_1\|_{L^{p'}(\mathbf{S})}\int_{\mathcal{N}}|k_1(x)|\,dx.$$

It remains to prove that $k_1 \in L^1(\mathcal{N})$. To this end, note that $k_1(x) = k_2(x)\phi_2(\sigma(x))$, where

$$k_2(x) = k(x)(1+|\sigma(x)|)^{\varepsilon}, \qquad \phi_2(s) = (1+|s|)^{-2\varepsilon},$$

where, by (18), k_2 and ϕ_2 also satisfy the assumptions of Lemma 6. Therefore,

$$\int_{\mathcal{N}}|k_1(x)|\,dx = \int_{\mathcal{N}}|k_2(x)\phi_2(\sigma(x))|\,dx \le C_1\|\phi_2\|_{L^{p'}(\mathbf{S})},$$

which completes the proof.

It follows from (8) and (9) that for every $f \in L^p(\mathcal{N})$ the limit

$$\lim_{N\to-\infty, M\to\infty} cf + f \star K_{N,M} \tag{19}$$

exists in L^p-norm and defines a bounded operator $Tf = f * T$ on $L^p(\mathcal{N})$.

Let π be an induced unitary representation of $\mathcal{N}$ as defined by (6). Since $K_{N,M}$ are compactly supported $L^1(\mathcal{N})$-functions, the estimates (8) and (9) combined with the transference principle of Coifman-Weiss [5] imply that for every $p \in (1,\infty)$ there exists a constant C_p such that

$$\|\pi_{K_{N,M}}\|_{L^p(\mathbf{S})\to L^p(\mathbf{S})} \le C_p. \tag{20}$$

Moreover, the limit

$$\lim_{N\to-\infty, M\to\infty} \pi_{c\delta_0+K_{N,M}} f \tag{21}$$

exists in the $L^p(\mathbf{S})$-norm for $f \in L^p(\mathbf{S})$ and defines a bounded operator on $L^p(\mathbf{S})$ denoted by π_T.

Proposition 8 *Let $T = c\delta_0 + K$, $T' = c'\delta_0 + K'$ belong to A_q. Then*

$$\pi_T \pi_{T'} f = \pi_{T'\star T} f \quad \textit{for } f \in L^p(\mathbf{S}). \tag{22}$$

PROOF. There is no loss of generality in assuming that $c = c' = 0$. Note that the left and the right-hand side of (22) define bounded operators on $L^p(\mathbf{S})$. Hence, it suffices to prove (22) on a dense set of $L^p(\mathbf{S})$. This will be done if we show that for $\varphi, \psi \in \mathcal{S}(\mathcal{N})$ and $f, g \in \mathcal{S}(\mathbf{S})$ one has

$$\langle \pi_T \pi_{T'} \pi_\varphi f, \pi_\psi g\rangle = \langle \pi_{T'\star T} \pi_\varphi f, \pi_\psi g\rangle. \tag{23}$$

Set $T' \star T = c''\delta_0 + K''$. By Theorem 2, $T' \star T \in A_q$. Then

$$\langle \pi_T \pi_{T'} \pi_\varphi f, \pi_\psi g\rangle = \lim_{N\to-\infty, M\to\infty} \langle \pi_{\varphi\star K'_{N,M}\star K_{N,M}\star\widetilde{\psi}} f, g\rangle, \tag{24}$$

$$\langle \pi_{T'\star T} \pi_\varphi f, \pi_\psi g\rangle = \lim_{N\to-\infty, M\to\infty} \langle \pi_{\varphi\star K''_{N,M}\star\widetilde{\psi}+c''\varphi\star\widetilde{\psi}} f, g\rangle, \tag{25}$$

where $\widetilde{\psi}(a) = \psi(a^{-1})$. Observe that

$$\varphi \star K'_{N,M} \star K_{N,M} \star \widetilde{\psi} \quad \text{and} \quad \varphi \star K''_{N,M} \star \widetilde{\psi} + c''\varphi \star \widetilde{\psi}$$

converge pointwise to $\varphi\star T'\star T\star\widetilde{\psi}$. Therefore, by applying Propositions 5 and 7, as well as the Lebesgue dominated convergence theorem, we get (23).

Proposition 9 *Let R and Q be kernels of order $r > 0$. In addition, let R be regular and let R^{-1} be a regular kernel of order $-r$ such that $R^{-1} \star R = R \star R^{-1} = \delta_0$. Then $Q \star R^{-1}$ is a kernel of order 0, and for every $f \in \mathcal{S}(\mathbf{S})$ we have*

$$\pi_{QR^{-1}} \pi_R f = \pi_Q f. \tag{26}$$

PROOF. Let φ be a compactly supported smooth function on $\mathcal{N}$ such that

$$\int_{\mathcal{N}} \varphi(a)\, da = 1.$$

Set $\varphi_t(a) = t^{-D}\varphi(\delta_{t^{-1}}a)$. Then, for $f \in \mathcal{S}(\mathbf{S})$, the functions $\pi_{\varphi_t} f$ converge in $\mathcal{S}(\mathbf{S})$ to f as $t \to 0$. Hence $\lim_{t\to0} \pi_T \pi_{\varphi_t} f = \pi_T f$ in $L^p(\mathbf{S})$ norm for $T = R, Q$. Moreover,

$$\begin{aligned}\pi_{QR^{-1}}\pi_R\pi_{\varphi_t}f &= \pi_{QR^{-1}}\pi_{\varphi_t\star R}f = \pi_{(\varphi_t\star R)\star(QR^{-1})}f\\ &= \pi_{\varphi_t\star Q}f = \pi_Q\pi_{\varphi_t}f.\end{aligned}$$

Taking the limit as t tends to 0, we obtain (26).

4 Maximal Estimates

Denote by $1 = d_1 < d_2 < \cdots < d_m$ the exponents of homogeneity of dilations on $\mathcal{N}$. Then

$$\mathcal{N} = \bigoplus_{j=1}^{m} \mathcal{N}_j,$$

where

$$\mathcal{N}_j = \{x \in \mathcal{N} : \delta_t x = t^{d_j} x,\ t > 0\}.$$

The subspace $\widetilde{\mathcal{N}} = \bigoplus_{j=1}^{m-1} \mathcal{N}_j$ is a homogenous linear complement to $\mathcal{N}_m$. It has been explained in Section 1 that $\widetilde{\mathcal{N}}$ may be identified with the quotient group $\mathcal{N}/\mathcal{N}_m$, and $\sigma : \mathcal{N} \to \widetilde{\mathcal{N}}$ with the corresponding quotient homomorphism.

For $\lambda \in \mathcal{N}_m^\star$, let

$$\pi^\lambda = \pi^{(\mathcal{N}_m,\lambda)}.$$

Then, as is easily seen, the right-regular representation ρ of $\mathcal{N}$ decomposes as

$$\rho_a f(x) = \int_{\mathcal{N}_m^\star} e^{2\pi i\langle \nu(x),\lambda\rangle}\pi_a^\lambda f^\lambda(\sigma(x))\, d\lambda, \tag{27}$$

where

$$f^\lambda(\sigma(x)) = \int_{\mathcal{N}_m} f(v\sigma(x))e^{-2\pi i\langle v,\lambda\rangle}\, dv$$

for $f \in C_c^\infty(\mathcal{N})$.

If T is a kernel of order $r > 0$ on $\mathcal{N}$, then

$$\langle \widetilde{T}, f\rangle = \langle T, f\circ\sigma\rangle$$

defines a kernel $\widetilde{T}$ of order r on $\widetilde{\mathcal{N}}$ such that

$$\widetilde{\rho}_{\widetilde{T}} = \pi_T^0,$$

where $\widetilde{\rho}$ is the right-regular representation of $\widetilde{\mathcal{N}}$.

Lemma 10 *Let T be a regular kernel of order 0 on a homogeneous subgroup M of $\mathcal{N}$. Then for every unitary representation π of $\mathcal{N}$, the operator π_T is bounded.*

For the proof it is sufficient to remark that any unitary representation of $\mathcal{N}$ restricted to M is also a unitary representation on the same Hilbert space.

Recall that a kernel T of order $r > 0$ is said to satisfy *the Rockland condition*, if for every nontrivial irreducible unitary representation π of $\mathcal{N}$ the operator π_T is injective on the domain of its closure (cf. Section 1). The pivotal point of the whole of our consideration here are the following two estimates (see Theorems 11 and 12 below).

Theorem 11 *Let M_1 and M_2 be homogeneous subgroups of $\mathcal{N}$. Assume that $M_1 \cup M_2$ generates the whole of $\mathcal{N}$. Let Q_1 and Q_2 be positive regular kernels of order $\ell > 0$ on M_1 and M_2 respectively such that $Q = Q_1 + Q_2$ satisfies the Rockland condition. Then for every symmetric kernel H of order ℓ, there exists a constant C such that*

$$\|Hf\|_{L^2(\mathcal{N})} \leq C\|(Q_1 + Q_2)f\|_{L^2(\mathcal{N})}, \qquad f \in C_c^\infty(\mathcal{N}). \tag{28}$$

PROOF. As a matter of fact, the proof is implicitly contained in [11] and [13]. For the convenience of the reader we will indicate how it can be made more explicit.

Suppose first that $\mathcal{N}$ is Abelian. Then the Fourier transform $\widehat{Q}$ of the tempered distribution Q is a continuous function on $\mathcal{N}^\star$ which is homogeneous of degree ℓ and does not vanishes except at the origin. Therefore, there exists a constant $C > 0$ such that

$$|\widehat{Q}(\xi)| \geq C|\xi|^\ell, \qquad \xi \in \mathcal{N}^\star,$$

which implies the assertion of the theorem in the Abelian case.

Now we proceed by induction. We assume that our assertion holds true for $\widetilde{\mathcal{N}} = \mathcal{N}/\mathcal{N}_m$. Once we prove that it holds for $\mathcal{N}$ as well, our proof will be completed.

Let H be a symmetric kernel of order $s = r\ell$. By the induction hypothesis,

$$\|\pi_H^0 f\|_{L^2(\widetilde{\mathcal{N}})} \leq C_0\|\pi_Q^0 f\|_{L^2(\widetilde{\mathcal{N}})}, \qquad f \in C_c^\infty(\widetilde{\mathcal{N}}).$$

The first important step is to extend this initial estimate to

$$\|\pi_H^\lambda f\|_{L^2(\widetilde{\mathcal{N}})} \leq C\Big(\|\pi_Q^\lambda f\|_{L^2(\widetilde{\mathcal{N}})} + \|f\|_{L^2(\widetilde{\mathcal{N}})}\Big), \qquad f \in C_c^\infty(\widetilde{\mathcal{N}}).$$

To this end one can imitate the proof of Theorem 3.19 of [11], where Lemma 3.18 of [11] is replaced by Lemma 10.

From now on we may follow the course of the proof of Theorem 3.1 of [13], where we take advantage of the homogeneity of the kernels in question and the Rockland condition, until we reach the final estimate

$$\|\pi_H^\lambda f\|_{L^2(\widetilde{\mathcal{N}})} \leq C\|\pi_Q^\lambda f\|_{L^2(\widetilde{\mathcal{N}})}$$

valid for all $\lambda \in \mathcal{N}_m^\star$ with the same constant $C > 0$. From here our assertion follows by (27).

Theorem 12 *Let R be a symmetric regular kernel of order $\ell > 0$ such that for every nontrivial irreducible unitary representation π of $\mathcal{N}$, the operator π_R is injective on the space $C^\infty(\pi)$. Then for every symmetric kernel H of order $s \leq \ell$, there exists a constant $C > 0$ such that*

$$\|Hf\|_{L^2(\mathcal{N})} \leq C\Big(\|Rf\|_{L^2(\mathcal{N})} + \|f\|_{L^2(\mathcal{N})}\Big), \qquad f \in C_c^\infty(\mathcal{N}). \tag{29}$$

Moreover, if $s = \ell$ then

$$\|Hf\|_{L^2(\mathcal{N})} \leq C\|Rf\|_{L^2(\mathcal{N})}, \qquad f \in C_c^\infty(\mathcal{N}). \tag{30}$$

PROOF. The proof is very similar to that of Theorem 11 but slightly simpler.

Corollary 13 *Let R be as in Theorem 12. Then R and all its positive integer powers R^N satisfy the Rockland condition.*

Let R be as in Theorem 12 and positive definite. Since $-R$ is essentially self-adjoint, it generates a continuous convolution semigroup of bounded operators P_t on $L^2(\mathcal{N})$. By Theorem (12), its corollary, and the argument of the proof of Theorem 1.13 of [7], $P_t f = f \star p_t$, where $p_t(x) = t^{-D/\ell} p_1(\delta_{t^{-1/\ell}} x)$ are smooth functions satisfying the estimates

$$|\partial p_1(x)| \leq C_\partial (1 + |x|)^{-D-\ell-|\partial|}$$

for every differential operator ∂ homogeneous of order $|\partial|$. The last estimate implies that for every $0 < \alpha < 1$

$$\langle R^\alpha, f\rangle = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-1-\alpha} \langle \delta_0 - p_t, f\rangle \, dt$$

defines a regular kernel of order $\alpha\ell$ which also satisfies the Rockland condition. Similarly, if $0 < \ell < D$, then

$$\langle R^{-1}, f\rangle = \int_0^\infty \langle p_t, f\rangle \, dt \tag{31}$$

is a regular kernel of order $-\ell$ such that $R \star R^{-1} = R^{-1} \star R = \delta_0$.

5 Proof of the main theorem

Let

$$\delta_t x = (t^{m_1}x_1, t^{m_2}x_2, ..., t^{m_d}x_d), \qquad x = (x_1, x_2, ..., x_d),$$

be a family of dilations on $\mathbf{R}^d$, where $m_1, m_2, ..., m_d$ are positive integers. Let $P(x)$ be a homogeneous of degree r polynomial such that

$$P(ix) > 0 \quad \text{for } x \neq 0.$$

Let $V(x) = \sum_{\beta \leq \gamma} c_\beta x^\beta \geq 0$. We define a nilpotent Lie algebra $\mathfrak{g}$ as follows (cf. [8], [6]). As a vector space $\mathfrak{g}$ is generated by the linearly independent vectors

$$\{X_1, \dots X_d, Y^{[\beta]} : 0 \leq \beta \leq \gamma\}$$

whose nontrivial commutators are

$$[X_k, Y^{[\beta]}] = \begin{cases} Y^{[\beta - e_k]} & \text{if } \beta - e_k \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{32}$$

where e_k is the d-tuple consisting of zeros except for a 1 in the kth position. The detailed discussion of the form of irreducible unitary representations of G can be found in [6].

Let $\{\delta_t\}_{t>0}$ be the one-parameter group of authomorphic dilations on $\mathfrak{g}$ determined by

$$\delta_t X_i = t^{m_i} X_i, \qquad \delta_t Y^{[\gamma]} = t^r Y^{[\gamma]}.$$

If we regard $\mathfrak{g}$ as a nilpotent Lie group G with multiplication given by the Campbell-Hausdorff formula, then the dilations δ_t are also automorphisms of the group structure on G. The Lie algebra $\mathfrak{g}$ of the Lie group G is identified with that of the left-invariant vector fields.

Let $\mathbf{S}$ and $\mathbf{V}$ denote the spans of X_j's and $Y^{[\beta]}$'s respectively. Then $\mathbf{S} = \mathbf{R}^d$ and $\mathbf{V}$ are Abelian subgroups of G invariant under δ_t, and $\mathbf{V}$ is a normal subgroup of G. Every element a of G can be uniquely written as $a = \nu(a)\sigma(a)$ (cf. Section 2), where $\nu(a) \in \mathbf{V}$, $\sigma(a) \in \mathbf{S}$ are polynomial mappings from G to $\mathbf{V}$ and $\mathbf{S}$ respectively. It is not difficult to check that if $a_1 = (v_1, x_1), a_2 = (v_2, x_2) \in G = \mathbf{V} \times \mathbf{S}$ then $\sigma(a_1 a_2) = x_1 + x_2$.

Let

$$Q_1 = P(X_1, X_2, ..., X_d), \qquad Q_2 = \left((Y^{[\gamma]})^*(Y^{[\gamma]})\right)^{1/2}, \qquad Q_3 = -iY^{[\gamma]}. \tag{33}$$

Of course, Q_1 and Q_3 are regular kernels of order r, whereas Q_2 is accretive.

Moreover, Q_2 is regular on the one-dimensional subgroup of G generated by $Y^{[\gamma]}$.

By the same argument as in Section 2 of [8], we conclude that there exist a functional ω on $\mathbf{V}$ and a regular symmetric kernel R of order r that satisfies the Rockland condition with the properties that

$$\pi^{\omega}_{Q_1} f = P(D)f, \qquad \pi^{\omega}_{Q_2} f = \pi^{\omega}_{Q_3} f = Vf. \tag{34}$$

$$\langle f \star R, \overline{f} \rangle \geq 0 \quad \text{for } f \in \mathcal{S}(G), \tag{35}$$

$$\pi^{\omega}_{R} f = (P(D) + V)f. \tag{36}$$

The construction of R presented in [8] is based on ideas of [6] and Theorem 2 of Hebisch [15].

We shall need a lemma.

Lemma 14 *The kernel $Q = Q_1^{\ell} + Q_2^{\ell}$ satisfies the Rockland condition for every $\ell > 0$.*

PROOF. Let π be a nontrivial irredicible unitary representation of G and $\overline{\pi}_Q \xi = 0$ for some ξ in the domain of $\overline{\pi}^{\ell}_Q$. Then ξ sits in the intersection of the domains of $\overline{\pi}_{Q_1^{\ell/2}}$ and $\overline{\pi}_{Q_2^{\ell/2}}$, and

$$\overline{\pi}_{Q_1^{\ell/2}} \xi = 0 = \overline{\pi}_{Q_2^{\ell/2}} \xi,$$

whence

$$\overline{\pi}_{Q_1} \xi = 0, \qquad \overline{\pi}_{Q_2} \xi = 0.$$

The latter operator is injective unless it is zero in which case the representation π is one-dimensional and corresponds to a character $X \mapsto e^{i\langle X, w\rangle}$ for some $w \neq 0$. Then, however, $\pi_Q = \pi_{Q_1}$ is just multiplication by $P(iw) > 0$, which proves our case.

Proof of Theorem 1. Fix $\alpha > 0$. There exist a nilpotent Lie group G of homogeneous dimension D, a functional ω, and a regular symmetric kernel R of order r that satisfies the Rockland condition such that (33) - (36) hold. We may always construct the group G in such a way that $D > r\alpha$.

The distribution R^{α} is a regular kernel of order $\ell = r\alpha$ satisfying the Rockland condition so, by (31), there exists a regular kernel $R^{-\alpha}$ of order $-r\alpha$ such that $R^{\alpha} \star R^{-\alpha} = R^{-\alpha} \star R^{\alpha} = \delta_0$. Note that the kernel $R^{-\alpha}$ is locally in $L^q(G)$ for $1 \leq q < D/(D - r\alpha)$.

It follows from (30) that the homogeneous of degree 0 operators

$$Q_1^\alpha R^{-\alpha} f = f \star R^{-\alpha} \star Q_1^\alpha, \qquad Q_2^\alpha R^{-\alpha} f = f \star R^{-\alpha} \star Q_2^\alpha, \tag{37}$$

are bounded on $L^2(G)$. One can check that their convolution kernels

$$R^{-\alpha} \star Q_1^\alpha \ \text{ and } \ R^{-\alpha} \star Q_2^\alpha$$

are principal value distributions that coincide with locally L^q-functions away from the origin for every $1 \leq q < D/(D - r\alpha)$. Therefore, by Theorem 2, for every $1 < p < \infty$ there exists a constant C such that

$$\|Q_1^\alpha R^{-\alpha} f\|_{L^p(G)} + \|Q_2^\alpha R^{-\alpha} f\|_{L^p(G)} \leq C\|f\|_{L^p(G)}. \tag{38}$$

Hence, by the transference principle of Coifman-Weiss (see Section 3),

$$\|\pi^\omega_{Q_j^\alpha R^{-\alpha}} f\|_{L^p(\mathbf{S})} \leq C\|f\|_{L^p(\mathbf{S})}, \quad j = 1, 2. \tag{39}$$

Applying (26) we obtain

$$\|\pi^\omega_{Q_j^\alpha} f\|_{L^p(\mathbf{S})} \leq C\|\pi^\omega_{R^\alpha} f\|_{L^p(\mathbf{S})} \quad \text{for } j = 1, 2, \tag{40}$$

which, by (34) and (36), gives (4).

We already know that the operator $T = (Q_1^\alpha + Q_2^\alpha)R^{-\alpha}$ is bounded on $L^p(G)$ for $1 < p < \infty$ and belongs to A_q for some $1 < q < \infty$. The Theorem 11 implies that it is also invertible on $L^2(G)$. Hence, by Theorems 3 and 2, T is invertible on $L^p(G)$ for every $1 < p < \infty$, and its inverse T^{-1} belongs to A_q. Therefore, by Proposition 8 and the transference principle, for every $1 < p < \infty$ there exists a constant $C > 0$ such that

$$\begin{aligned} \|f\|_{L^p(\mathbf{S})} &= \|\pi^\omega_{T^{-1}} \pi^\omega_{(Q_1^\alpha + Q_2^\alpha)R^{-\alpha}} f\|_{L^p(\mathbf{S})} \\ &\leq C\|\pi^\omega_{(Q_1^\alpha + Q_2^\alpha)R^{-\alpha}} f\|_{L^p(\mathbf{S})}. \end{aligned} \tag{41}$$

By application of (26), (34), and (36), we obtain (5).

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