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MELIN CALCULUS FOR GENERAL HOMOGENEOUS GROUPS

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Abstract. The purpose of this note is to give an extension of the symbolic calculus of Melin for convolution operators on nilpotent Lie groups with dilations. Whereas the calculus of Melin is restricted to stratified nilpotent groups, the extension offered here is valid for general homogeneous groups. Another improvement concerns the L^2-boundedness theorem, where our assumptions on the symbol are relaxed. The zero-class conditions that we require are of the type

$$|D^\alpha a(\xi)| \leq C_\alpha \prod_{j=1}^{R} \rho_j(\xi)^{-|\alpha_j|},$$

where ρ_j are "partial homogeneous norms" depending on the variables ξ_k for $k > j$ in the natural grading of the Lie algebra (and its dual) determined by dilations. Finally, the class of admissible weights for our calculus is substantially broader. Let us also emphasize the relative simplicity of our argument if compared to that of Melin.

Introduction

The purpose of this note is to give an extension of the symbolic calculus of Melin [7] for convolution operators on nilpotent Lie groups with dilations. The calculus can be viewed as a higher order generalization of the Weyl calculus for pseudodifferential operators of Hörmander [3]. In fact, the idea of such a calculus is very similar. It consists in describing the product

$$a\#b = (a^\vee \star b^\vee)^\wedge, \qquad a, b \in C_c^\infty(\mathfrak{g}^\star),$$

on a homogeneous Lie group G, where $f^\wedge$ and $f^\vee$ denote the Abelian Fourier transforms on the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^\star$, and its continuity in terms of suitable norms similar to those used in the theory of pseudodifferential operators. An integral part of the calculus is a L^2-boundedness theorem of the Calderón-Vaillancourt type.

This has been done by Melin whose starting point was the following formula

$$a\#b(\xi) = \mathbf{U}(a \otimes b)F(\xi, \xi),$$

where

$$\mathbf{U}(F)^{\vee}(x,y) = F^{\vee}(\frac{x-y+xy}{2}, \frac{y-x+xy}{2}), \qquad x,y \in \mathfrak{g}.$$

Melin shows that the unitary operator $\mathbf{U}$ can be imbedded in a one-parameter unitary group U_t with the infinitesimal generator Γ which is a differential operator on $\mathfrak{g}^\star \times \mathfrak{g}^\star$ with polynomial coefficients, and he thoroughly investigates the properties of Γ under the assumption that G is a homogeneous stratified group. From the continuity of $\mathbf{U}$ he derives a composition formula for classes of symbols satisfying the estimates

$$|D^\alpha a(\xi)| \le C_\alpha (1 + |\xi|)^{m-|\alpha|}, \tag{0.1}$$

where $|\cdot|$ is the homogeneous norm on $\mathfrak{g}^\star$ and $|\alpha|$ is a homogeneous length of a multiindex α. He also proves an L^2-boundedness theorem for symbols satisfying (0.1) with $m = 0$.

Our extension goes in various directions. First of all the calculus of Melin is restricted to stratified nilpotent groups, whereas the extension offered here is valid for general homogeneous groups. Another improvement concerns the L^2-boundedness theorem, where our assumptions on the symbol are less restrictive. The zero-class conditions that we require are

$$|D^\alpha a(\xi)| \le C_\alpha \prod_{j=1}^{R} \rho_j(\xi)^{-|\alpha_j|},$$

where ρ_j are "partial homogeneous norms" depending on the variables ξ_k for $k > j$ in the natural grading of the Lie algebra (and its dual) determined by dilations, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_R)$ is the corresponding representation of the multiindex α relative to the grading. This direction of generalization of the boundedness theorem had been suggested by Howe [5] even before the Melin calculus was created. Finally, the class of admissible weights for our calculus is substantially broader. Let us also emphasize the relative simplicity of our argument if compared to that of Melin.

Most of the techniques applied here have been already developed in a very similar context of [2]. They heavily rely on the methods of the Weyl calculus of Hörmander [3]. We take this opportunity to clarify some technical points which remained somewhat obscure in [2]. One major mistake is also corrected. Some repetition is therefore unavoidable. In [2] the reader will also find more on the background and history of various symbolic calculi on nilpotent Lie groups.

1. Preliminaries

Let X be a finite dimensional Euclidean space. Denote by $< \cdot, \cdot >$ and $\| \cdot \|$ the scalar product and the corresponding Euclidean norm. These are fixed throughout the paper. Whenever we identify $X^\star$ with

X, it is by means of the duality determined by the scalar product. Let $X = \bigoplus_{k=1}^{R} X_k$ be an orthogonal sum. Fix an orthonormal basis $\{e_{kj}\}_{j=1}^{n_k}$ in X_k, where $n_k = \dim X_k$. Thus the variable $x \in X$ splits into $x = (x_1, x_2, \ldots, x_R)$, where

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{kn_k}) \in X_k.$$

The *length* of a multiindex

$$\alpha = (\alpha_k)_{k=1}^{R} = (\alpha_{kj}) \in \mathbf{N}^{\dim X}.$$

is defined by

$$|\alpha| = \sum_{k=1}^{R} |\alpha_k|, \qquad |\alpha_k| = \sum_{j=1}^{n_k} \alpha_{kj}.$$

Let

$$D_{kj} f(x) = f'(x) e_{kj},$$

and

$$D^\alpha = D_1^{\alpha_1} \ldots D_R^{\alpha_R}, \qquad D_k^{\alpha_k} = D_{k1}^{\alpha_{k1}} \ldots D_{kn_k}^{\alpha_{kn_k}}.$$

We assume that X is endowed with a family of dilations

$$\delta_t x_k = t^{d_k}, \qquad x_j \in X_k,$$

with eigenvalues $\mathcal{D} = \{d_k\}$, where

$$1 \le d_1 \le d_2 \le \cdots \le d_R.$$

The *homogeneous length* of a multiindex α is defined by

$$d(\alpha) = \sum_{k=1}^{R} d_k |\alpha_k|.$$

Let

$$|x| = \sum_{k=1}^{R} \|x_k\|^{1/d_k}$$

be the corresponding homogeneous norm. For $0 \le k \le R$ let $|x|_{R+1} = 0$ and

$$|x|_k = \sum_{j=k}^{R} \|x_j\|^{1/d_j}, \qquad 1 \le k \le R.$$

Let

$$\mathbf{q}_x(z)^2 = \sum_{k=1}^{R} \frac{\|z_k\|^2}{q_k(x)^{2d_k}} \qquad x \in X,$$

where $q_k(x) = 1 + |x|_{k+1}$, be a family of norms (a Riemannian metric) on X. More generally, let $\mathcal{G}(X)$ denote the set of all metrics $\mathbf{g}$ of the form

$$\mathbf{g}_x(z)^2 = \sum_{k=1}^{R} \frac{\|z_k\|^2}{g_k(x)^{2d_k}},$$

where

$$g_k(x) = \delta + |x|_{k+1}, \qquad \delta > 0, \quad 0 \le k \le R.$$

Lemma 1.1. *Let* $\mathbf{g} \in \mathcal{G}(X)$*. For every* $0 \le k \le R$*,*

$$\frac{1}{2} \le \frac{g_k(x)}{g_k(y)} \le 2 \quad \textit{if} \quad \mathbf{g}_x(x-y) < \frac{1}{2R}.$$

Proof. Observe that $\mathbf{g}_x(x-y) < \frac{1}{2R}$ yields

$$\|x_j - y_j\|^{1/d_j} \le \frac{g_k(x)}{2R}, \qquad k < j \le R,$$

so

$$\sum_{j=k+1}^{R} \|x_j - y_j\| \le \frac{1}{2} g_k(x),$$

and consequently

$$g_k(x) \le g_k(y) + \sum_{j=k+1}^{R} \|x_j - y_j\| \le g_k(y) + \frac{1}{2} g_k(x),$$
$$g_k(y) \le g_k(x) + \sum_{j=k+1}^{R} \|x_j - y_j\| \le \frac{3}{2} g_k(x),$$

which implies

$$\frac{1}{2} \le \frac{g_k(x)}{g_k(y)} \le 2.$$

□

Lemma 1.2. *Let* $\mathbf{g} \in \mathcal{G}(X)$*. There exist constants* $C, M > 0$ *independent of* $\mathbf{g}$ *such that, for every* $0 \le k \le R$*,*

$$g_k(x) \le C g_k(y) \Big(1 + \mathbf{g}_y(x-y)\Big) \tag{1.3}$$

and

$$g_k(x) \le C g_k(y) \Big(1 + \mathbf{g}_x(x-y)\Big)^M. \tag{1.4}$$

Proof. We start with inequality (1.3). We have

$$\begin{aligned} g_k(x) &\le g_k(y) + \sum_{j=k+1}^{R} \|x_j - y_j\|^{1/d_j} \\ &\le g_k(y)\left(1 + \frac{\sum_{j=k+1}^{R} \|x_j - y_j\|^{1/d_j}}{g_k(y)}\right) \\ &\le g_k(y)\left(1 + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|^{1/d_j}}{g_j(y)}\right) \\ &\le g_k(y)\left(R + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|}{g_j(y)^{d_j}}\right) \\ &\le C g_k(y)\Big(1 + \mathbf{g}_y(x-y)\Big). \end{aligned}$$

The second inequality is proved by induction. In fact, if $k = R$, there is nothing to prove. Assume (1.4) holds for $k+1$ with some constants $C, M > 0$. Then

$$\begin{aligned} g_k(x) &\le g_k(y) + \sum_{j=k+1}^{R} \|x_j - y_j\|^{1/d_j} \\ &\le g_k(y)\Big(1 + \frac{\sum_{j=k+1}^{R} \|x_j - y_j\|^{1/d_j}}{g_k(y)}\Big) \\ &\le g_k(y)\Big(1 + \frac{g_{k+1}(x)}{g_{k+1}(y)} \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|^{1/d_j}}{g_{k+1}(x)}\Big). \end{aligned}$$

By induction hypothesis,

$$\begin{aligned} g_k(x) &\le C g_k(y)(1 + g_x(x-y))^M \left(1 + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|^{1/d_j}}{g_{k+1}(x)}\right) \\ &\le C g_k(y)(1 + g_x(x-y))^M \left(R + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|}{g_j(y)^{d_j}}\right) \\ &\le C_1 g_k(y)(1 + \mathbf{g}_x(x-y))^{M+1}, \end{aligned}$$

which shows that (1.4) holds also for k with new constants C_1 and $M_1 = M + 1$. ∎

A family of Euclidean norms (a metric) $\mathbf{g} = \{\mathbf{g}_x\}_{x\in X}$ on X is called *slowly varying* if there exists $0 < \gamma \le 1$ such that

$$\gamma \le \frac{\mathbf{g}_y}{\mathbf{g}_x} \le \frac{1}{\gamma}, \quad \text{if} \quad \mathbf{g}_x(x-y) < \gamma. \tag{1.5}$$

Two metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ are said to be *equivalent* if there exists a constant $C > 0$ such that

$$C^{-1}\mathbf{g}_1 \le \mathbf{g}_2 \le C\mathbf{g}_1. \tag{1.6}$$

A metric $\mathbf{g}$ on X is called *tempered* with respect to another metric $\mathbf{G}$, or briefly $\mathbf{G}$-tempered, if there exist $C, M > 0$ such that

$$\left\{\frac{\mathbf{g}_x}{\mathbf{g}_y}\right\}^{\pm 1} \le C\Big(1 + \mathbf{G}_x(x-y)\Big)^M, \qquad \mathbf{g}_x \le \mathbf{G}_x. \tag{1.7}$$

Note that a self-tempered metric is automaticly slowly varying.

Corollary 1.8. *All metrics* $\mathbf{g} \in \mathcal{G}(\mathfrak{g})$ *are slowly varying and uniformly self-tempered.*

Proof. That all metrics in $\mathcal{G}(X)$ are uniformly self-tempered and therefore slowly varying follows immediately from Lemma 1.2. Alternatively, one can invoke Lemma 1.1 to show that they are slowly varying. □

Lemma 1.9. *If* $\mathbf{g}$ *is a self-tempered family of norms with constants* C, M*, then for every* $x, y, z \in X$

$$1 + \mathbf{g}_x(x-y) \le C\Big((1 + \mathbf{g}_y(x-y)\Big)^{M+1}, \tag{1.10}$$

$$1 + \mathbf{g}_x(x-y) \le C(1 + \mathbf{g}_z(x-z))^M(1 + \mathbf{g}_z(z-y)), \tag{1.11}$$

$$1 + \mathbf{g}_x(x-y) \le C^2\Big(1 + \mathbf{g}_x(x-z)\Big)^M\Big(1 + \mathbf{g}_y(z-y)\Big)^{M+1}, \tag{1.12}$$

Proof. In fact,

$$\begin{aligned} 1 + \mathbf{g}_x(x-y) &\le 1 + C\mathbf{g}_y(x-y)\Big(1 + \mathbf{g}_y(x-y)\Big)^M \\ &\le C\Big(1 + \mathbf{g}_y(x-y)\Big)^{M+1}, \end{aligned}$$

as required in (1.10). Moreover, by (1.10),

$$\begin{aligned} 1+\mathbf{g}_x(x-y) &\le 1 + \mathbf{g}_x(x-z) + \mathbf{g}_x(z-y) \\ &\le 1 + \mathbf{g}_x(x-z) + C\mathbf{g}_z(z-y)\Big(1 + \mathbf{g}_x(x-z)\Big)^M \\ &\le \Big(1 + \mathbf{g}_x(x-z)\Big)^M\Big(1 + C\mathbf{g}_z(z-y)\Big) \end{aligned}$$

which gives (1.11). Finally, (1.10) and (1.11) imply (1.12). □

A strictly positive function $\mathbf{m}$ on X is *a* $\mathbf{G}$*-tempered weight* on X with respect to the $\mathbf{G}$-tempered metric $\mathbf{g}$, if it satisfies the conditions

$$\left\{\frac{\mathbf{m}(x)}{\mathbf{m}(y)}\right\}^{\pm 1} \le C \quad \text{if} \quad \mathbf{g}_x(x-y) \le \gamma \tag{1.13}$$

and

$$\left\{\frac{\mathbf{m}(x)}{\mathbf{m}(y)}\right\}^{\pm 1} \le C\Big(1 + \mathbf{G}_x(x-y)\Big)^M \tag{1.14}$$

for some $C, M > 0$. The weights form a group under multiplication. A typical example of a weight for $\mathbf{g} \in \mathcal{G}(\mathfrak{g})$ is $\mathbf{m}(x) = 1 + |x|_k$. A universal example is

$$\mathbf{m}(x) = 1 + \mathbf{g}_x(x - x_0), \tag{1.15}$$

where x_0 is fixed. Note also that the constant function $\mathbf{1}(x) = 1$ is a weight for every metric $\mathbf{g}$.

Let $\mathbf{m}$ be a weight with respect to a metric $\mathbf{g}$. For $f \in C^\infty(X)$ let

$$|f|^{\mathbf{m}}_{(k)}(\mathbf{g}) = \sup_{x \in X} \frac{\mathbf{g}_x(D^k f(x))}{\mathbf{m}(x)},$$

and

$$|f|^{\mathbf{m}}_{k}(\mathbf{g}) = \max_{0 \le j \le k} |f|^{\mathbf{m}}_{(j)}(\mathbf{g}),$$

where D stands for the Fréchet derivative, and

$$\mathbf{g}_x(D^k f(x)) = \sup_{\mathbf{g}_x(y_j) \le 1} |D^k f(x)(y_1, y_2, \dots, y_k)|.$$

Let

$$S^{\mathbf{m}}(X, \mathbf{g}) = \{a \in C^\infty(X) : |a|^{\mathbf{m}}_k(\mathbf{g}) < \infty, \text{ all } k \in \mathbf{N}\}.$$

$S^{\mathbf{m}}(X, \mathbf{g})$ is a Fréchet space with the family of seminorms $|\cdot|^{\mathbf{m}}_k(\mathbf{g})$. Thus $f \in C^\infty(X)$ belongs to $S^{\mathbf{m}}(X, \mathbf{g})$ if and only if it satisfies the estimates

$$|D^\alpha f(x)| \le C_\alpha \mathbf{m}(x) \prod_{k=1}^{R} g_k(x)^{-d_k|\alpha_k|},$$

where $\alpha = (\alpha_1, \dots, \alpha_R)$. Arbitrary seminorms in $S^{\mathbf{m}}(X, \mathbf{g})$ will be denoted by $|\cdot|^{\mathbf{m}}_{\mathbf{g}}$.

Apart from the Fréchet topology in the spaces $S^{\mathbf{m}}(X, \mathbf{g})$ it is convenient to introduce a *weak topology* of the C^∞-convergence on Fréchet bounded subsets. By the Ascoli theorem, this is equivalent to the pointwise convergence of bounded sequences in $S^{\mathbf{m}}$. Following Manchon [6] we call a mapping $T : S^{\mathbf{m}_1} \to S^{\mathbf{m}_2}$ *double-continuous*, if it is both Fréchet continuous and weakly continuous. Moreover, $C^\infty_c(X)$ is weakly dense in $S^{\mathbf{m}}(X, \mathbf{g})$. The last assertion is a consequence of Proposition 2.1 b) below.

2. The method of Hörmander

The following construction of a partition of unity is due to Hörmander [3]. Also the lemma that follows is an important principle of the Hörmander theory. For the convenience of the reader we include the proofs here.

Proposition 2.1. *Let* $\mathbf{g}$ *be a slowly varying metric on* X*.*

a) *For every* $0 < r < \gamma$ *there exists a sequence* $x_\nu \in X$ *such that* X *is the union of the balls*

$$B_\nu = B_\nu(r) = \{x \in X : \mathbf{g}_{x_\nu}(x - x_\nu) < r\}$$

and no point $x \in X$ *belongs to more than* N *balls, where* N *does not depend on* x*.*

b) *There exists a family of functions* $\phi_\nu \in C_c^\infty(B_\nu)$ *bounded in* $S^{\mathbf{1}}(X, \mathbf{g})$ *and such that*

$$\sum_\nu \phi_\nu(x) = 1, \qquad x \in X.$$

c) *For* $x \in X$ *let*

$$d_\nu(x) = \mathbf{g}_{x_\nu}(x - x_\nu).$$

If the metric $\mathbf{g}$ *is self-tempered, then there exist constants* $M, C_0 > 0$ *such that*

$$\sum_\nu \Big(1 + d_\nu(x)\Big)^{-M} \leq C_0, \qquad x \in X.$$

All the estimates in the construction depend just on the constant γ *in (1.5), constants* C, M *in* (1.7)*, and the choice of* r*.*

Proof. a) Let $0 < r < \gamma$. Let $\{x_\nu\}$ be a maximal sequence of points in X such that

$$\mathbf{g}_{x_\nu}(x_\mu - x_\nu \geq \gamma r, \qquad \mu \neq \nu.$$

Let $x \in X$. Note that

$$\mathbf{g}_x(x - x_\nu) < \gamma r \quad \text{implies} \quad \mathbf{g}_{x_\nu}(x - x_\nu) < r.$$

Therefore, either $\mathbf{g}_{x_\nu}(x - x_\nu) < r$ for some ν, or

$$\mathbf{g}_x(x - x_\nu) \geq \gamma r \quad \text{and} \quad \mathbf{g}_{x_\nu}(x - x_\nu) \geq r \geq \gamma r.$$

The latter contradicts the maximality of our sequence. The former implies that $X \subset \bigcup_\nu B_\nu$.

To show that the covering is uniformly locally finite suppose that $x \in B_\nu$. Then $\mathbf{g}_{x_\nu}(x - x_\nu) < r$, which implies $\mathbf{g}_x(x - x_\nu) < r\gamma < 1$. On the other hand $\mathbf{g}_x(x_\mu - x_\nu) \geq \gamma r$ for $\mu \neq \nu$. The number of points from a uniformly discrete set in a unit ball is bounded independently of the given norm $\mathbf{g}_x$ so we are done.

b) Let $0 < r < r_1 < \gamma$. Let $\psi \in C_c^\infty(-r_1^2, r_1^2)$ be equal to 1 on the smaller interval $[-r^2, r^2]$. If

$$\psi_\nu(x) = \psi(\mathbf{g}_{x_\nu}(x - x_\nu)^2),$$

then, by part a), $\sum_\mu \psi_\mu(x) \geq 1$ for every $x \in X$, and it is not hard to see that

$$\phi_\nu(x) = \frac{\psi_\nu(x)}{\sum_\mu \psi_\mu}$$

has all the required properties.

c) Let $r < \gamma$. Let $x \in X$. For $k \in \mathbf{N}$ let

$$M_k = \{\nu : d_\nu(x) < k\}.$$

It is sufficient to show that the number $|M_k|$ of the elements in M_k is bounded by a polynomial in k. Let $\nu \in M_k$ and let

$$V_\nu = \{z \in X : \mathbf{g}_x(z - x_\nu) < r_k\},$$

where

$$r_k = \frac{r}{C(1+k)^M}.$$

Observe that V_ν is contained both in B_ν (see part a)) and in the ball

$$V = \{z \in X : \mathbf{g}_x(z - x) < R_k\},$$

where $R_k = r_k + C(1+k)^{M+1}$. In fact, if $\mathbf{g}_x(z - x_\nu) < r_k$, then

$$\mathbf{g}_{x_\nu}(z - x_\nu) \le C\mathbf{g}_x(z - x_\nu)(1+k)^M < r,$$

and

$$\begin{aligned}\mathbf{g}_x(z - x) &\le \mathbf{g}_x(z - x_\nu) + \mathbf{g}_x(x_\nu - x)\\ &< r_k + C\mathbf{g}_{x_\nu}(x_\nu - x)\Big(1 + \mathbf{g}_{x_\nu}(x_\nu - x)\Big)^M\\ &< r_k + C\Big(1 + 1 + \mathbf{g}_{x_\nu}(x_\nu - x)\Big)^{M+1} < r_k + C(1+k)^{M+1}\end{aligned}$$

Hence

$$C_1|M_k|r_k^{\dim X} \le \sum_{\nu \in M_k} |V_\nu| \le N|\bigcup_{\nu \in M_k} V_\nu| \le N|V| \le C_1 N R_k^{\dim X},$$

which immediately implies the desired estimate

$$|M_k| \le N\Big(1 + \frac{R_k}{r_k}\Big)^{\dim X}.$$

□

Lemma 2.2. *Let X be a finite dimensional vector space with a Euclidean norm $\|\cdot\|$. Let $r_1 > r > 0$. Let L be an affine function such that $L(x) \ne 0$ for $x \in B(x_0, r_1)$. Then for every $k \in \mathbf{N}$,*

$$\|D^k \frac{1}{L}(x)\| \le \frac{k! r_1}{(r_1 - r)^{k+1}|L(x_0)|}, \qquad x \in B(x_0, r).$$

The estimate does not depend on the choice the norm.

Proof. We may assume that $x_0 = 0$ and $L(0) = 1$. Let ξ be a linear functional on X such that $L(x) = \langle \xi, x\rangle + 1$. Since

$$L(x) = \langle \xi, x\rangle + 1 > 0, \qquad \|x\| < r_1,$$

it follows that $\|\xi\| \le \frac{1}{r_1}$ and

$$L(x) \ge 1 - \frac{r}{r_1}, \qquad x \in B(0, r).$$

Consequently,

$$\left\| D^k \frac{1}{L}(x) \right\| \le \frac{k! \|\xi\|^k}{|L(x)|^{k+1}} \le \frac{k! \left(\frac{1}{r_1}\right)^k}{\left(\frac{r_1 - r}{r_1}\right)^{k+1}}$$
$$\le \frac{k! r_1}{(r_1 - r)^{k+1}}$$

for x in $B(0,r)$. □

For the general theory of slowly varying metrics and its applications to the theory of pseudodifferential calculus the reader is referred to Hörmander [4], vol. I and III.

3. Homogeneous groups

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a fixed scalar product. The dual vector space $\mathfrak{g}^\star$ will be identified with $\mathfrak{g}$ by means of the scalar product. We shall also regard $\mathfrak{g}$ as a Lie group with the Campbell-Hausdorff multiplication

$$x_1 \circ x_2 = x_1 + x_2 + r(x_1, x_2),$$

where

$$r(x_1, x_2) = \frac{1}{2}[x_1, x_2] + \frac{1}{12}([x_1, [x_1, x_2]] + [x_2, [x_2, x_1]])$$
$$+ \frac{1}{24}[x_2, [x_1, [x_2, x_1]]] + \dots$$

is the (finite) sum of terms of order at least 2 in the Campbell-Hausdorff series for $\mathfrak{g}$.

The Lebesue measure dx is a biinvariant Haar measure for the group $\mathfrak{g}$. The formula for convolution reads

$$f \star g(x) = \int_{\mathbf{G}} f(x \circ y^{-1}) g(y)\, dy, \qquad f, g \in L^1(\mathfrak{g}).$$

Let $\{\delta_t\}_{t>0}$, be a family of group dilations on $\mathfrak{g}$ and let

$$\mathfrak{g}_k = \{x \in \mathfrak{g} : \delta_t x = t^{d_k} x\}, \qquad 1 \le k \le R,$$

where $1 \le d_1 \le d_2 \le \dots \le d_R$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_R \tag{3.1}$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \begin{cases} \mathfrak{g}_k, & \text{if } d_i + d_j = d_k, \\ \{0\}, & \text{if } d_i + d_j \notin \mathcal{D}, \end{cases}$$

where $\mathcal{D} = \{d_j : 1 \le j \le R\}$. Let $x \to |x|$ be the homogeneous norm on $\mathfrak{g}$ as defined in Section 1. All remaining notation of Section 1 holds as well. Observe that $\mathfrak{g}$ is a Euclidean space so we can define the usual Fourier transforms

$$\widehat{f}(y) = \int_{\mathfrak{g}} e^{-i<x,y>}\,dx, \qquad f^{\vee}(x) = \int_{\mathfrak{g}} e^{i<x,y>} f(y)\,dx,$$

and adjust the Lebesgue measure so that

$$\int_{\mathfrak{g}} |f(x)|^2\,dx = \int_{\mathfrak{g}} |\widehat{f}(y)|^2\,dy, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

For $f, g \in \mathcal{S}(\mathfrak{g})$, we define

$$f\#g(y) = (f^{\vee} \star g^{\vee})^{\wedge}(y), \qquad y \in \mathfrak{g}.$$

4. The Melin operator

For a function $f \in C_c^{\infty}(\mathfrak{g} \times \mathfrak{g})$ let

$$\mathbf{U}f(\mathbf{y}) = \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i<\mathbf{x},\mathbf{y}>} f^{\vee}(\mathbf{x}) e^{-i<r(\mathbf{x}),\widetilde{\mathbf{y}}>} d\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathfrak{g} \times \mathfrak{g}$, and $\widetilde{\mathbf{y}} = \frac{y_1 + y_2}{2}$. We shall refer to $\mathbf{U}$ as the *Melin operator* on $\mathfrak{g}$. The importance of $\mathbf{U}$ consists in

$$\widehat{f \star g}\,(y) = \mathbf{U}(\widehat{f} \otimes \widehat{g})(y, y), \qquad y \in \mathfrak{g}, \tag{4.1}$$

which is checked directly. By an easy induction, we get

Lemma 4.2. *For every $f \in C_c^{\infty}(\mathfrak{g} \times \mathfrak{g})$,*

$$D_1^{\alpha} D_2^{\beta} \mathbf{U}f(y_1, y_2) = \sum_{d(\gamma)+d(\delta)=d(\alpha)+d(\beta)} c_{\gamma\delta} \mathbf{U}(D_1^{\gamma} D_2^{\delta} f)(y_1, y_2),$$

where $c_{\gamma\delta} \in \boldsymbol{C}$.

Let

$$\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{R-1} \tag{4.3}$$

The commutator

$$\mathfrak{g}' \times \mathfrak{g}' \ni (x_1, x_2) \to [x_1, x_2]' \in \mathfrak{g}',$$

where $'$ stands for the orthogonal projection onto $\mathfrak{g}'$, makes $\mathfrak{g}'$ into a Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{g}_R$ with $x \to x'$ playing the role of the canonical quotient homomorphism. The group multiplication in $\mathfrak{g}'$ is

$$x_1 \circ' x_2 = x_1 + x_2 + r(x_1, x_2)'.$$

Proposition 4.4. *For $f \in C_c^{\infty}(\mathfrak{g} \times \mathfrak{g})$,*

$$\mathbf{U}f(\mathbf{y}, \lambda) = \mathbf{U}'\Big(P_{\lambda} f(\cdot, \lambda)\Big)(\mathbf{y}), \qquad \mathbf{y} \in \mathfrak{g}', \lambda \in \mathfrak{g}_R, \tag{4.5}$$

where

$$P_{\lambda} g(\mathbf{y}) = \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i<\mathbf{x},\mathbf{y}>} g^{\vee}(\mathbf{x}) e^{-i<r(\mathbf{x}),\widetilde{\lambda}>} d\mathbf{x}, \quad g \in C_c^{\infty}(\mathfrak{g}' \times \mathfrak{g}'),$$

is an integral operator on $C_c^\infty(\mathfrak{g}')$ invariant under Abelian translations, and $\mathbf{U}'$ stands for the Melin operator on $\mathfrak{g}'$.

Proof. In fact,

$$\begin{aligned}\mathbf{U}f(\mathbf{y},\lambda) &= \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i<\mathbf{x},\mathbf{y}>}e^{-i<t,\lambda>}f^\vee(\mathbf{x},t)e^{-i<r'(\mathbf{x}),\widetilde{\mathbf{y}}>}e^{-i<r(\mathbf{x}),\widetilde{\lambda}>}d\mathbf{x}dt \\ &= \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i<\mathbf{x},\mathbf{y}>}\Big(f(\mathbf{x}^\vee,\lambda)e^{-i<r(\mathbf{x}),\widetilde{\lambda}>}\Big)e^{-i<r'(\mathbf{x}),\widetilde{\mathbf{y}}>}d\mathbf{x} \\ &= \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-<\mathbf{x},\mathbf{y}>}(P_\lambda f(\cdot,\lambda))^\vee(\mathbf{x})e^{-i<r'(\mathbf{x}),\widetilde{\mathbf{y}}>}d\mathbf{x} \\ &= \mathbf{U}'\Big(P_\lambda f(\cdot,\lambda)\Big)(\mathbf{y})\end{aligned}$$

for all $f\in C_c^\infty(\mathfrak{g}\times\mathfrak{g})$, $\mathbf{y}\in\mathfrak{g}'\times\mathfrak{g}'$, $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. □

For the background on homogeneous groups we recommend Folland-Stein [1].

5. The inductive step

In what follows we apply the notation of Section 1 among others to $X=\mathfrak{g}$ and $X=\mathfrak{g}\times\mathfrak{g}$. In the latter case we employ the product norm $\|\mathbf{x}\|^2=\|x_1\|^2+\|x_2\|^2$, the product dilations $\delta_t\mathbf{x}=\delta_t x_1+\delta_t x_2$, and the product homogeneous norm $|\mathbf{x}|=|x_1|+|x_2|$.

From now on we focus on the self-tempered metric $\mathbf{q}$. This metric is in a way maximal for other $\mathbf{q}$-tempered metrics we are going to consider in Theorem 6.4 below. However, the induction we are going to make requires that we consider metrics

$$\mathbf{q}_{\mathbf{x}}^\lambda(\mathbf{y})=\mathbf{q}_{(\mathbf{x},\lambda)}(\mathbf{y},0)$$

on $\mathfrak{g}'$ which are different from $\mathbf{q}'=\mathbf{q}^0$, the counterpart of $\mathbf{q}$ on $\mathfrak{g}'$. Therefore, for the sake of flexibility, we begin with a metric $\mathbf{g}\in\mathcal{G}(\mathfrak{g})$. Then

$$\begin{aligned}\mathbf{G}_{\mathbf{x}}(\mathbf{z})^2 &= (\mathbf{g}\oplus\mathbf{g})_{\mathbf{x}}(\mathbf{z})^2=\mathbf{g}_{x_1}(z_1)^2+\mathbf{g}_{x_2}(z_2)^2 \\ &= \sum_{j=1}^R\frac{\|z_{1j}\|^2}{g_j(x_1)^{2d_j}}+\sum_{j=1}^R\frac{\|z_{2j}\|^2}{g_j(x_2)^{2d_j}},\end{aligned}$$

where

$$\mathbf{z}=(z_1,z_2)=(z_{11},\dots,z_{1R}\mid z_{21},\dots,z_{2R}),$$

is a metric in $\mathcal{G}(\mathfrak{g}\times\mathfrak{g})$.

Let $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$ (see (3.1) and (4.3)). It is easily seen that the family of metrics

$$\mathbf{G}_{\mathbf{x}}^\lambda(\mathbf{y})=\mathbf{G}_{(\mathbf{x},\lambda)}(\mathbf{y},0),\qquad \mathbf{x},\mathbf{y}\in\mathfrak{g}'\times\mathfrak{g}',$$

is uniformly slowly varying and uniformly self-tempered. One just has to observe that each of the metrics $\mathbf{G}^\lambda$ is equivalent to a metric in $\mathcal{G}(\mathfrak{g})$ (see (1.6)) with a constant C independent of λ. Let γ be a joint constant for all the metrics $\mathbf{G}^\lambda$. Let $g_j^\lambda(\mathbf{x}) = g_j(\mathbf{x},\lambda)$. Let $B_\nu^\lambda = B_\nu^\lambda(\mathbf{x}_\nu^\lambda, r) \subset \mathfrak{g}' \times \mathfrak{g}'$ be the covering of Proposition 2.1 for $\mathbf{G}^\lambda$. (To simplify notation we shall write B_ν for B_ν^λ and x_ν for x_ν^λ.) Let

$$d_\nu^\lambda(\mathbf{y}) = \mathbf{G}^\lambda_{\mathbf{x}_\nu}(\mathbf{y} - \mathbf{x}_\nu)$$

(cf. Proposition 2.1).

Here comes the crucial step in our argument.

Lemma 5.1. *Let* $\mathbf{g} \in \mathcal{G}(\mathfrak{g})$ *and* $\mathbf{G} = \mathbf{g} \oplus \mathbf{g}$. *For every* N, *there exist* C *and* k *such that*

$$|P_\lambda f(\mathbf{y})| \le C|f|^{\mathbf{1}}_k(\mathbf{G}^\lambda)\Big(1 + d_\nu^\lambda(\mathbf{y})\Big)^{-N} \tag{5.2}$$

for $f \in C_c^\infty(B_\nu^\lambda)$ *uniformly in* $\lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$ *and* $\nu \in \mathbf{N}$.

Proof. Let $f \in C_c^\infty(\mathfrak{g}' \times \mathfrak{g}')$ be supported in B_ν. There exist C and k such that

$$\begin{aligned} |P_\lambda f(\mathbf{y})| &\le \iint_{\mathfrak{g}'\times\mathfrak{g}'} |f^\vee(\mathbf{x})| d\mathbf{x} = \|f\|_{A(\mathfrak{g}'\times\mathfrak{g}')} \\ &= \|f_\lambda\|_{A(\mathfrak{g}'\times\mathfrak{g}')} \le C|f|^{\mathbf{1}}_k(\mathbf{G}^\lambda), \end{aligned} \tag{5.3}$$

where

$$f_\lambda(\mathbf{y}) = f\Big(g_1^\lambda(\mathbf{x}_\nu)^{d_1}\mathbf{y}_1, \dots g_{R-1}^\lambda(\mathbf{x}_\nu)^{d_{R-1}}\mathbf{y}_{R-1}\Big),$$

and $\|\cdot\|_{A(\mathfrak{g}'\times\mathfrak{g}')}$ stands for the Fourier algebra norm. The last inequality is achieved by the Sobolev inequality

$$\|f\|_{A(\mathfrak{g}'\times\mathfrak{g}')} \le C(s) \sum_{|\alpha|\le s} \|D^\alpha f\|_2$$

applied to f_λ which is supported in a ball of radius 1 with respect to the norm $\|\cdot\|$.

Assume now that (5.2) is true for some N. Let $d_\nu^\lambda(\mathbf{y}) = a > 1$. Note that otherwise the estimate is a matter of course. Therefore there exists $\xi \in (\mathfrak{g}' \times \mathfrak{g}')^\star$ of unit length with respect to the norm dual to $\mathbf{G}^\lambda_{\mathbf{x}_\nu}$ such that $\xi(\mathbf{y} - \mathbf{x}) \ge ca$ for $\mathbf{x} \in B(\mathbf{x}_\nu, r_1)$, where $0 < r < r_1 < \gamma$ and $c > 0$. The norm one condition reads

$$\begin{aligned} 1 = (\mathbf{G}^\lambda_{\mathbf{x}_\nu})^\star(\xi)^2 &= \sum_{1\le j\le R-1} (g_j)^\lambda(\mathbf{x}_\nu)^{2d_j}\, \|\xi_j\|^2 \\ &\ge \sum_{1\le j\le R-1} (1 + \|\lambda\|^{\frac{1}{d_R}})^{2d_j}\, \|\xi_j\|^2. \end{aligned}$$

Then $L(\mathbf{x}) = \langle \mathbf{x} - \mathbf{y}, \xi\rangle$ does not vanish on $B(\mathbf{x}_\nu, r_1)$ so, by Lemma 2.2,

$$\mathbf{G}^\lambda_{\mathbf{x}_\nu}\Big(D^k L^{-1}(\mathbf{x})\Big) \le \frac{C_k}{a}, \qquad \mathbf{x} \in B(\mathbf{x}_\nu, r).$$

Note that $L(\mathbf{y}) = 0$. Therefore,

$$P_\lambda(Lf)(\mathbf{y}) = [P_\lambda, L]f(\mathbf{y}) = \sum_{j=1}^{R-1} \xi_j (1 + \|\lambda\|^{1/d_R})^{d_j} P_\lambda(f_{\lambda,j})(\mathbf{y}),$$

where

$$f_{\lambda,j}(\mathbf{z}) = \frac{1}{i}(1 + \|\lambda\|^{1/d_R})^{d_R - d_j} \langle r_j(iD), \frac{\widetilde{\lambda}}{(1 + \|\lambda\|^{1/d_R})^{d_R}} \rangle f(\mathbf{z}),$$

and

$$r_j(\mathbf{x}, \lambda) = D_{(x_1)_j} r(\mathbf{x}, \lambda) + D_{(x_2)_j} r(\mathbf{x}, \lambda)$$

is a homogeneous polynomial of degree $d_R - d_j$. Thus $f_{\lambda,j}$ are uniformly bounded in $S^{\mathbf{1}}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{G}^\lambda)$. It follows that

$$|P_\lambda(f)(\mathbf{y})| \le \Big(\sum_{j=1}^{R-1} |P_\lambda\Big((L^{-1} f)_{\lambda,j} \Big)(\mathbf{y})|^2 \Big)^{1/2},$$

and consequently, by Lemma 2.2 and induction hypothesis,

$$\begin{aligned} |P_\lambda f(\mathbf{y})| &\le \frac{C_k}{a} |f|_k^{\mathbf{1}}(\mathbf{G}^\lambda) \Big(1 + d_\nu^\lambda(\mathbf{y}) \Big)^{-N} \\ &\le C_k' |f|_k^{\mathbf{1}}(\mathbf{G}^\lambda) \Big(1 + d_\nu^\lambda(\mathbf{y}) \Big)^{-N-1}, \end{aligned}$$

which completes the proof of (5.2). □

We continue with metrics $\mathbf{g} \in \mathcal{G}(\mathfrak{g})$ and $\mathbf{G} = \mathbf{g} \oplus \mathbf{g} \in \mathcal{G}(\mathfrak{g} \times \mathfrak{g})$. Let $\mathbf{m}$ be a $\mathbf{G}$-weight. Then $\mathbf{m}_\lambda(\mathbf{x}) = \mathbf{m}(\mathbf{x}, \lambda)$ is a weight on $\mathfrak{g}' \times \mathfrak{g}'$ with respect to $\mathbf{G}^\lambda$ (which is self-tempered), and the family of weights is uniform in λ. Let $\phi_\nu^\lambda \in C_c^\infty(B_\nu)$ be the partition of unity of Proposition 2.1 on $\mathfrak{g}' \times \mathfrak{g}'$ for $\mathbf{G}^\lambda$. By Proposition 2.1, ϕ_ν^λ are bounded in $S^{\mathbf{1}}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{G}^\lambda)$ uniformly in ν and λ.

Observe that

$$\begin{aligned} \mathbf{m}_\lambda(\mathbf{y}) &\le C_1 \mathbf{m}_\lambda(\mathbf{x}_\nu) \Big(1 + \mathbf{G}^\lambda_{\mathbf{x}_\nu}(\mathbf{y} - \mathbf{x}_\nu) \Big)^M \\ &\le C_1 \mathbf{m}_\lambda(\mathbf{x}_\nu) \Big(1 + d_\nu^\lambda(\mathbf{y}) \Big)^M. \end{aligned} \tag{5.4}$$

Proposition 5.5. *For every λ, there exists a unique double-continuous extension of P_λ to a mapping*

$$P_\lambda : S^{\mathbf{m}_\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{G}^\lambda) \to S^{\mathbf{m}_\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{G}^\lambda).$$

All the estimates hold uniformly in λ.

Proof. Note that

$$\boldsymbol{n}_\lambda(\mathbf{y}) = \mathbf{m}^\lambda(\mathbf{y})^{-1} \prod_{j=1}^{R-1} g_j(y_1)^{d_j|\alpha_j|} g_j(y_2)^{d_j|\beta_j|}$$

is a product of weights so a weight itself. P_λ commutes with translations so, by (5.4) and Lemma 5.1,

$$\begin{aligned}
&\boldsymbol{n}_\lambda(\mathbf{y})^{-1}|D^\alpha_{y_1}D^\beta_{y_2}P_\lambda(\phi^\lambda_\nu f)(\mathbf{y})| \\
&\qquad \le C_1\boldsymbol{n}_\lambda(\mathbf{x}_\nu)^{-1}\Big(1+d^\lambda_\nu(\mathbf{y})\Big)^M |P_\lambda\Big(D^\alpha D^\beta(\phi^\lambda_\nu f)\Big)(\mathbf{y})| \\
&\qquad \le C_2|f|^{\mathbf{m}_\lambda}_{k+|\alpha|+|\beta|}\Big(1+d^\lambda_\nu(\mathbf{y})\Big)^{-N+M}.
\end{aligned}$$

Let N be so large that

$$\sum_\nu \Big(1+d^\lambda_\nu(\mathbf{y})\Big)^{-N+M} < \infty,$$

see Proposition 2.1, c). Then our estimate which remains valid for f in a bounded subset of $S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{G}^\lambda)$ without any restriction on support implies that for every $\mathbf{y}\in\mathfrak{g}'$

$$f \to \sum_\nu P_\lambda(\phi^\lambda_\nu f)(\mathbf{y})$$

defines a weakly continuous linear form on $S^{\mathbf{m}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{G}^\lambda)$. Consequently, P_λ admits an extension to the whole of $S^{\mathbf{m}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{G}^\lambda)$, and

$$\begin{aligned}
|D^\alpha_{y_1}D^\beta_{y_2}P_\lambda(f)(\mathbf{y})| &= |\sum_\nu P_\lambda(\phi^\lambda_\nu f)(\mathbf{y})| \\
&\le C|f|^{\mathbf{m}_\lambda}_{k+|\alpha|+|\beta|}\mathbf{m}_\lambda(\mathbf{y})\prod_{j=1}^{R-1} g_j(y_1)^{-d_j\alpha_j}g_j(y_2)^{-d_j\beta_j},
\end{aligned}$$

for $f\in S^{\mathbf{m}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{G}^\lambda)$, which shows that P_λ is both Fréchet and weakly continuous.

□

6. Symbolic calculus

Recall from Section 4 that the Melin operator $\mathbf{U}$ has been defined for $f\in C^\infty_c(\mathfrak{g}\times\mathfrak{g})$.

Theorem 6.1. *Let $\mathbf{g}\in\mathcal{G}(\mathfrak{g})$. Let $\mathbf{G}=\mathbf{g}\oplus\mathbf{g}$ and let $\mathbf{m}$ be a $\mathbf{G}$-weight on $\mathfrak{g}\times\mathfrak{g}$. There exists a double-continuous extension of the Melin operator to*

$$\mathbf{U}: S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{G})\to S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{G}).$$

Proof. Suppose that $\mathfrak{g}$ is as in (3.1) and proceed by induction. If $R=1$, $\mathfrak{g}$ is Abelian and $\mathbf{U}=I$ so the assertion is obvious. Assume that our theorem is true for $\mathfrak{g}'$ as in (4.3) and $\mathbf{U}=\mathbf{U}'$. For $\lambda\in\mathfrak{g}_R$ and $f\in S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{q})$ let $f_\lambda(\mathbf{y})=f(\mathbf{y},\lambda)$, $q^\lambda_j(\mathbf{y})=q_j(\mathbf{y},\lambda)$, and $\mathbf{m}_\lambda(\mathbf{y})=\mathbf{m}(\mathbf{y},\lambda)$.

By hypothesis, $f_\lambda = f(\cdot,\lambda) \in S^{\mathbf{m}_\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{q}^\lambda)$ uniformly in λ (cf. the previous section). Now Proposition 5.5 yields

$$P_\lambda f_\lambda \in S^{\mathbf{m}_\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{q}^\lambda)$$

uniformly in λ so, by the induction hypothesis,

$$\mathbf{U}' P_\lambda f_\lambda \in S^{\mathbf{m}_\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{q}^\lambda)$$

uniformly in λ. The same holds true for the derivatives

$$\Big(q_R(\lambda_s)\frac{\partial}{\partial \lambda_s}\Big)^j \mathbf{U}' P_\lambda f_\lambda,$$

where $\lambda = (\lambda_1, \lambda_2)$ and $s = 1, 2$, which is checked directly. Thus, by (4.5), we get the desired estimate: For every $k_1 \in \mathbf{N}$, there exists $k_2 \in \mathbf{N}$ such that

$$|\mathbf{U}f|^{\mathbf{m}}_{k_1}(\mathbf{G}) \le C|f|^{\mathbf{m}}_{k_2}(\mathbf{G}), \qquad f \in S^{\mathbf{m}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{G}).$$

Finally, by induction hypothesis and Proposition 5.5, $\mathbf{U}$ is also weakly continuous. This completes the proof. □

Corollary 6.2. *Let* $\mathbf{m}_1$, $\mathbf{m}_2$ *be* $\mathbf{q}$*-weights on* $\mathfrak{g}$*. Then*

$$C_c^\infty(\mathfrak{g}) \times C_c^\infty(\mathfrak{g}) \ni (a, b) \to a\#b \in \mathcal{S}(\mathfrak{g})$$

extends uniquely to a double-continuous mapping

$$S^{\mathbf{m}_1}(\mathfrak{g}, \mathbf{q}) \times S^{\mathbf{m}_2}(\mathfrak{g}, \mathbf{q}) \to S^{\mathbf{m}_1\mathbf{m}_2}(\mathfrak{g}, \mathbf{q}).$$

Proof. This is a straightforward consequence of (4.1) and Theorem 6.1 applied to the metric $\mathbf{Q} = \mathbf{q} \oplus \mathbf{q}$ on $\mathfrak{g} \times \mathfrak{g}$. □

We are ready now to deal with general $\mathbf{q}$-tempered metrics. Let $\mathbf{g}$ be a $\mathbf{q}$-tempered slowly varying metric on $\mathfrak{g}$. If $\mathbf{g} \le \mathbf{q}$, every $\mathbf{q}$-tempered $\mathbf{g}$-weight $\mathbf{m}$ is also a $\mathbf{q}$-weight and $S^{\mathbf{m}}(\mathbf{g}) \subset S^{\mathbf{m}}(\mathbf{q})$. The identity mapping $I : S^{\mathbf{m}}(\mathbf{g}) \to S^{\mathbf{m}}(\mathbf{q})$ is double-continuous. Let

$$\boldsymbol{n}_\alpha(x) = \prod_{j=1}^{R} g_j(x)^{d_j|\alpha_j|}.$$

Then $\boldsymbol{n}_\alpha$ is a $\mathbf{q}$-tempered $\mathbf{g}$-weight.

One more remark is in order. By Lemma 4.2,

$$D^\gamma(f\#g) = \sum_{d(\alpha)+d(\beta)=d(\gamma)} c_{\alpha\beta} D^\alpha f \# D^\beta g, \tag{6.3}$$

for $f, g \in \mathcal{S}(\mathfrak{g})$. By Corollary 6.2, the formula extends to f, g in the symbol classes governed by the metric $\mathbf{q}$.

Theorem 6.4. *Let* $\mathbf{g}$ *be a* $\mathbf{q}$*-tempered slowly varying metric on* $\mathfrak{g}$ *such that* $\mathbf{g} \le \mathbf{q}$*. Let* $\mathbf{m}_1, \mathbf{m}_2$ *be* $\mathbf{q}$*-tempered* $\mathbf{g}$*-weights. Then, for every* $a \in S^{\mathbf{m}_1}(\mathfrak{g}, \mathbf{g})$ *and every* $b \in S^{\mathbf{m}_2}(\mathfrak{g}, \mathbf{g})$*,* $a\#b \in S^{\mathbf{m}_1\mathbf{m}_2}(\mathfrak{g}, \mathbf{g})$ *and the mapping*

$$S^{\mathbf{m}_1}(\mathfrak{g}, \mathbf{g}) \times S^{\mathbf{m}_2}(\mathfrak{g}, \mathbf{g}) \ni (a, b) \to a\#b \in \mathcal{S}^{\mathbf{m}_1\mathbf{m}_2}(\mathfrak{g}, \mathbf{g})$$

is double-continuous.

Proof. It is sufficient to show that for every multiindex α, there exist seminorms $|\cdot|_{\mathbf{g}}^{\mathbf{m}_1}$ and $|\cdot|_{\mathbf{g}}^{\mathbf{m}_2}$ such that

$$\boldsymbol{n}_\alpha(x)\mathbf{m}_1^{-1}(x)\mathbf{m}_2^{-1}(x)|D^\alpha(a\#b)(x)| \leq |a|_{\mathbf{g}}^{\mathbf{m}_1}\cdot|b|_{\mathbf{g}}^{\mathbf{m}_2}.$$

Let us start with $\alpha = 0$. By Corollary 6.2, there exist seminorms $|\cdot|_{\mathbf{q}}^{\mathbf{m}_1}$ and $|\cdot|_{\mathbf{q}}^{\mathbf{m}_2}$, hence also seminorms $|\cdot|_{\mathbf{g}}^{m_1}$ and $|\cdot|_{\mathbf{g}}^{m_2}$ such that

$$\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|a\#b| \leq C|a|_{\mathbf{q}}^{\mathbf{m}_1}\cdot|b|_{\mathbf{q}}^{\mathbf{m}_2} \leq C_1|a|_{\mathbf{g}}^{\mathbf{m}_1}\cdot|b|_{\mathbf{g}}^{\mathbf{m}_2}.$$

Since $D^\alpha a \in S^{\mathbf{m}_1\boldsymbol{n}_\alpha^{-1}}(\mathfrak{g},\mathbf{g})$, $D^\beta b \in S^{\mathbf{m}_2\boldsymbol{n}_\beta^{-1}}(\mathfrak{g},\mathbf{g})$, the above gives

$$\boldsymbol{n}_\alpha\boldsymbol{n}_\beta\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|D^\alpha a\#D^\beta b| \leq C_1|D^\alpha a|_{\mathbf{g}}^{\mathbf{m}_1\boldsymbol{n}_\alpha^{-1}}\cdot|D^\beta b|_{\mathbf{g}}^{\mathbf{m}_2\boldsymbol{n}_\beta^{-1}}.$$

Note that the expression on the right is a product of two seminorms, denoted by $|a|_{\mathbf{g}}^{\mathbf{m}_1}$ and $|b|_{\mathbf{g}}^{\mathbf{m}_2}$, in $S^{\mathbf{m}_1}(\mathbf{g})$ and $S^{\mathbf{m}_2}(\mathbf{g})$ respectively. Thus,

$$\boldsymbol{n}_\alpha\boldsymbol{n}_\beta\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|D^\alpha a\#D^\beta b| \leq C_2|a|_{\mathbf{g}}^{\mathbf{m}_1}\cdot|b|_{\mathbf{g}}^{\mathbf{m}_2}, \tag{6.5}$$

and, by (6.3),

$$\begin{aligned}\boldsymbol{n}_\gamma\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|D^\gamma(a\#b)| &\leq \max_{|\alpha|+|\beta|=|\gamma}\boldsymbol{n}_\alpha\boldsymbol{n}_\beta\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|D^\gamma(a\#b)|\\ &\leq C\max_{|\alpha|+|\beta|=|\gamma|}\boldsymbol{n}_\alpha\boldsymbol{n}_\beta\mathbf{m}_1^{-1}\mathbf{m}_2^{-1}|D^\alpha a\#D^\beta b|,\end{aligned}$$

which combined with (6.5) completes the proof. □

Here are two important examples of metrics whose symbol spaces enjoy the above symbolic calculus. The first one is

$$\mathbf{g}_x(z)^2 = \sum_{j=1}^R \frac{\|z_j\|^2}{(1+|x|)^{2d_j}}.$$

Another one is

$$\mathbf{g}_x(z)^2 = \sum_{j=1}^R \frac{\|z_j\|^2}{(1+|x|_j)^{2d_j}}.$$

By Lemma 1.1 and Lemma 1.2 both metrics are slowly varying and $\mathbf{q}$-tempered. It is also clear that in both cases $\mathbf{g} \leq \mathbf{q}$.

7. L^2-BOUNDEDNESS

Let ϕ_ν the standard partition of unity for the metric $\mathbf{q}$ on $\mathfrak{g}$. Let $\Phi_{\mu\nu}(\mathbf{x}) = \phi_\mu(x_1)\phi_\nu(x_2)$, where $\mathbf{x} = (x_1, x_2) \in \mathfrak{g}\times\mathfrak{g}$. Let $\mathbf{Q} = \mathbf{q}\oplus\mathbf{q}$. Note that, by (1.11),

$$1+\mathbf{q}_{x_\nu}(x_\mu - x_\nu) \leq C\Big(1+\mathbf{q}_y(x_\mu - y)\Big)^M\Big(1+\mathbf{q}_y(x_\nu - y)\Big). \tag{7.1}$$

Corollary 7.2. *Let $f \in S^{\mathbf{1}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{Q})$. Let*

$$f_{\mu\nu}(y) = \mathbf{U}(\Phi_{\mu\nu} f)(y, y)$$

be a function on $\mathfrak{g}$. Then, for every N, there exists a seminorm $|\cdot|_{\mathbf{Q}}$ in $S^{\mathbf{1}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{Q})$ such that for every μ, ν,

$$\|f_{\mu\nu}\|_{A(\mathfrak{g})} \leq |f|_{\mathbf{Q}} \Big(1 + \mathbf{q}_{x_\nu}(x_\mu - x_\nu)\Big)^{-N}.$$

Proof. The function

$$\mathbf{m}_{\mu\nu}(\mathbf{y}) = \Big(1 + \mathbf{q}_{y_1}(x_\mu - y_1)\Big)^{-NM} \Big(1 + \mathbf{q}_{y_2}(x_\nu - y_2)\Big)^{-N},$$

where $\mathbf{y} = (y_1, y_2)$ and M is as in (7.1), is a $\mathbf{Q}$-tempered $\mathbf{Q}$-weight (see (1.15)). If $y_1 \in B_\nu$ and $y_2 \in B_\mu$, then

$$\mathbf{q}_{x_\mu}(y_1 - x_\mu) < \gamma \qquad \mathbf{q}_{x_\nu}(y_2 - x_\nu) < \gamma$$

so, by (1.5),

$$\mathbf{q}_{y_1}(y_1 - x_\mu) \leq 1, \qquad \mathbf{q}_{y_2}(y_2 - x_\nu) \leq 1,$$

which implies that $\mathbf{m}_{\mu\nu}^{-1}$ is uniformly bounded on the support of $\Phi_{\mu\nu}$. This implies that

$$|\Phi_{\mu\nu} f|_{\mathbf{Q}}^{\mathbf{m}_{\mu\nu}} \leq C |\Phi_{\mu\nu} f|_{\mathbf{Q}}^{\mathbf{1}}$$

with the same constant $C > 0$, for all μ and ν. Consequently, since $\Phi_{\mu\nu}$ is supported in $B_\mu \times B_\nu$, we have a trivial uniform estimate

$$\Phi_{\mu\nu} f \in S^{\mathbf{m}_{\mu\nu}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{Q}).$$

By Proposition 6.1,

$$\mathbf{U}(\Phi_{\mu\nu} f) \in S^{\mathbf{m}_{\mu\nu}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{Q}) \tag{7.3}$$

uniformly in μ, ν. Now, by (7.1),

$$\mathbf{m}_{\mu\nu}(y, y) \leq C\Big(1 + \mathbf{q}_{x_\nu}(x_\mu - x_\nu)\Big)^{-N},$$

and, by (7.3), for every k, there exists k_1 such that

$$|D_y^\alpha f_{\mu\nu}(y)| \leq C_1 |\Phi_{\mu\nu} f|_{k_1}^{\mathbf{m}_{\mu\nu}} \mathbf{m}_{\mu\nu}(y, y) \leq C_2 |f|_{k_1}^{\mathbf{1}} \Big(1 + \mathbf{q}_{x_\nu}(x_\mu - x_\nu)\Big)^{-N}$$

for $|\alpha| \leq k$. If k is large enough, our assertion follows by the Sobolev inequality. □

Theorem 7.4. *Let $a \in S^{\mathbf{1}}(\mathfrak{g}, \mathbf{q})$. The linear operator $f \to Af = f \star a^\vee$ defined initially on the dense subspace $C_c^\infty(\mathfrak{g})$ of $L^2(\mathfrak{g})$ extends to a bounded mapping of $L^2(\mathfrak{g})$. To be more specific, there exists a seminorm $|\cdot|_{\mathbf{q}}^{\mathbf{1}}$ in $S^{\mathbf{1}}(\mathfrak{g}, \mathbf{q})$ such that*

$$\|Af\|_{L^2(\mathfrak{g})} \leq |a|_{\mathbf{q}}^{\mathbf{1}} \|f\|_{L^2(\mathfrak{g})}, \qquad f \in C_c^\infty(\mathfrak{g}).$$

Proof. Let
$$A_\nu f = f \star (\phi_\nu a)^\vee, \qquad f \in L^2(\mathfrak{g}).$$
Since $\phi_\nu \in C_c^\infty(\mathfrak{g})$, the operators A_ν are bounded. Moreover, by (4.1) with the notation of Corollary 7.2,
$$A_\mu^\star A_\nu f(y) = (\overline{a} \otimes a)^\vee_{\mu\nu} \star f(y), \qquad A_\mu A_\nu^\star f(y) = (a \otimes \overline{a})^\vee_{\mu\nu} \star f,$$
so that, by Corollary (7.2),
$$\|A_\mu^\star A_\nu\| + \|A_\mu A_\nu^\star\| \leq \Big(|a|^{\mathbf{1}}_{\mathbf{q}}\Big)^2 \Big(1 + g_{x_\nu}(x_\nu - x_\mu)\Big)^{-N},$$
where N can be taken as large, as we wish, and $|\cdot|^{\mathbf{1}}_{\mathbf{q}}$ is a seminorm in $S^{\mathbf{1}}(\mathfrak{g}, \mathbf{q})$ depending only on N.

On the other hand,
$$a = \sum_u \phi_u a,$$
where the the series is weakly convergent in $S^{\mathbf{1}}(\mathfrak{g}, \mathbf{q})$ so that, by Corollary 6.2,
$$Af = \sum_\mu A_\mu f, \qquad f \in C_c^\infty(\mathfrak{g})$$
in the sense of weak convergence in $S^{\mathbf{1}}(\mathfrak{g}, \mathbf{q})$ of the Fourier transforms. Thus, the sequence of operators A_μ satisfies the hypothesis of Cotlar's Lemma (see e.g. Stein [8]) , and therefore the series $\sum_\mu A_\mu$ is strongly convergent to the extension of our operator A whose norm is bounded by $C|a|^{\mathbf{1}}_{\mathbf{q}}$ (see Proposition 2.1). □

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