1.10 Exercises

Families of sets

Recall that $r(\mathcal{F})$ denotes the smallest ring (=*pierścień*) containing the family $\mathcal{F}$, $a(\mathcal{F})$ is the corresponding algebra of sets, $s(\mathcal{F})$ is the *sigma*-ring and, $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$. The Polish term *ciało zbiorów* we translate as an algebra of sets (though sometimes a *field* is also used).

1.10.1 Let $\mathcal{R}$ be a ring of sets. Note that if $A, B \in \mathcal{R}$ then $A \triangle B \in \mathcal{R}$ and $A \cap B \in \mathcal{R}$. Check that $(\mathcal{R}, \triangle, \cap)$ is then a ring in the algebraic sense, in particular $\triangle$ is associative and distributive with respect to $\triangle$.

1.10.2 Let $\mathcal{F}$ be a family of subsets of X such that $X \in \mathcal{F}$ and $A \setminus B \in \mathcal{F}$ for $A, B \in \mathcal{F}$. Check that $\mathcal{F}$ is an algebra of sets.

1.10.3 Note that intersection of arbitrary many rings, algebras... is again a ring, algebra etc.

1.10.4 Note that if $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{P}(X)$ then $\alpha(\mathcal{F}) \subseteq \alpha(\mathcal{G})$; here α stands for any of the symbols r, s, a, σ.

1.10.5 Let $\mathcal{G}$ be a family of all finite subsets of X. Describe $r(\mathcal{G})$, $s(\mathcal{G})$, $a(\mathcal{G})$ and $\sigma(\mathcal{G})$.

1.10.6 Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets and let $Z \subseteq X$. Prove that

$$a(\mathcal{A} \cup \{Z\}) = \{(A \cap Z) \cup (B \cap Z^c) : \ A, B \in \mathcal{A}\}.$$

1.10.7 Check that if $\mathcal{C}$ is a family of subsets of X and $X = \bigcup_{n=1}^\infty C_n$ for some $C_n \in \mathcal{C}$ then $s(\mathcal{C}) = \sigma(\mathcal{C})$.

1.10.8 Note that if a family of sets is a ring and a monotone class then it is also a σ-ring.

1.10.9 Check that if $\mathcal{A}$ is an algebra of sets and $\mathcal{A}$ is closed under taking countable **disjoint** unions then $\mathcal{A}$ is a σ–algebra.

1.10.10 Let $\mathcal{A}$ be a finite algebra of sets. Prove that $|\mathcal{A}| = 2^n$ for some natural number n. Hint: figure out the value of n first.

1.10.11 Let $\mathcal{F}$ be a countable family of sets. Prove that the generated algebra $a(\mathcal{F})$ is also countable.

1.10.12 Prove that if $\mathcal{A}$ is an infinite σ–algebra then $\mathcal{A}$ has at least $\mathfrak{c}$ elements. Hint: Check first that an infinite σ-algebra contains a sequence of pairwise disjoint nonempty sets; use the fact that $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\mathbb{N})$.

Set functions

1.10.13 Let μ be a finitely additive set function defined on a ring $\mathcal{R}$. Check that for any $A, B, C \in \mathcal{R}$ we have

(i) $|\mu(A) - \mu(B)| \leqslant \mu(A \triangle B)$;
(ii) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$;
(iii) $\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B) - \mu(A \cap C) - \mu(B \cap C) + \mu(A \cap B \cap C)$.

Find analogous formulas for 4, 5... sets.

1.10.14 Given μ as above, check that $A \sim B \iff \mu(A \triangle B) = 0$ defines an equivalence relation on $\mathcal{R}$.

1.10.15 Let X be a finite set. Check that the formula $\mu(A) = \frac{|A|}{|X|}$ defines a probability measure on $\mathcal{P}(X)$.

1.10.16 Fix a sequence $(x_n) \subseteq X$ and a sequence (c_n) of nonnegative reals. Prove that

$$\mu(A) = \sum_{n: x_n \in A} c_n$$

defines a measure on $\mathcal{P}(X)$ (when in doubt, consider a finite sequence $x_1, \ldots, x_n$). When such a measure is finite?

1.10.17 Note that $\mathcal{P}(\mathbb{N})$ is a σ-algebra generated by all singletons. prove that every measure on $\mathcal{P}(\mathbb{N})$ is of the form described in the previous exercise.

1.10.18 Let μ be a measure on a σ-algebra $\mathcal{A}$ and let $A_n \in \mathcal{A}$. Assuming that $\mu(A_n \cap A_k) = 0$ for $n \neq k$, prove that

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

1.10.19 Complete the proof of Theorem 1.5.5 as follows: For a measure space (X, Σ, μ) define $\widehat{\Sigma}$ as the family of sets of the form $A \triangle N$, where $A \in \Sigma$, $N \subseteq B$ for some $B \in \Sigma$ of measure zero. Then $\widehat{\Sigma}$ is a σ-algebra and the formula $\widehat{\mu}(A \triangle N) = \mu(A)$ correctly defines an extension of μ to a measure on $\widehat{\Sigma}$.

On the real line; the Lebesgue measure

1.10.20 Let $\mathcal{R}$ be a ring of subsets of $\mathbb{R}$ generated by all the intervals of the form $[a, b)$. Check that $A \in \mathcal{R}$ if and only if A can be written as a union of pairwise disjoint such intervals.

1.10.21 Prove that the family $\mathbb{R}$ of sets of the form

$$(F_1 \cap V_1) \cup \ldots \cup (F_k \cap V_k),$$

where F_i are closed , V_i are open, $k \in \mathbb{N}$, is an algebra.

1.10.22 Check that σ–algebra $Bor(\mathbb{R})$ of Borel sets is generated by every of the following families

(i) open intervals with rational endpoints;

(ii) closed intervals;
(iii) half-lines of the form $(-\infty, a]$;
(iv) half-lines of the form (a, ∞);
(v) closed intervals with rational endpoints.

1.10.23 Check the following properties of the Lebesgue measure λ

(i) $\lambda(A) = 0$ for every finite A;
(ii) $\lambda[a, b] = \lambda(a, b) = b - a$ for $a < b$;
(iii) $\lambda(U) > 0$ for every open $U \neq \emptyset$;
(iv) $\lambda(A) = 0$ for any countable A.

1.10.24 Give examples of measurable sets $A \subseteq \mathbb{R}$ such that

(i) $\lambda(A) = 1$ and A is unbounded open set;
(ii) $\lambda(\mathrm{int}(A)) = 1$, $\lambda(A) = 2$, $\lambda(\overline{A}) = 3$;
(iii) $\lambda(A) = 0$ and $A \subseteq [0, 1]$ is uncountable.

Remark: $\mathrm{int}(A)$ is the interior of the set, that is the biggest open subset A.

1.10.25 Construct for a given $\varepsilon > 0$, a closed set $F \subseteq [0, 1]$ with empty interior such that $\lambda(F) > 1 - \varepsilon$.

I method: Modify the construction of the Cantor set.

II method: Let $(q_n)_n$ be a sequence of all the rational numbers from $[0, 1]$. Consider the open set $V = \bigcup_{n=1}^{\infty}(q_n - \varepsilon 2^{-n}, q_n + \varepsilon 2^{-n})$ for properly chosen $\varepsilon > 0$.

1.10.26 Note that for every Lebesgue measurable set $M \in \mathfrak{L}$, if $\lambda(M) < \infty$ then for every $\varepsilon > 0$ there is a bounded measurable set $M_0 \subseteq M$ such that $\lambda(M \setminus M_0) < \varepsilon$.

1.10.27 Note that there is a closed set $F \subseteq [0, 1]$ of positive measure consisting of irrationals.

1.10.28 For $B \subseteq \mathbb{R}$ and $x \neq 0$, xB denotes $\{xb : \ b \in B\}$ (a homothetic copy of B).

Check that a homothetic copy of an open set is open and that the family of those $B \in Bor(\mathbb{R})$ for which $xB \in Bor(\mathbb{R})$ for every $x \neq 0$ is a σ-algebra. Conclude that for every $B \in Bor(\mathbb{R})$ and x we have $xB \in Bor(\mathbb{R})$ (i.e. the σ-algebra $Bor(\mathbb{R})$ is homothetic invariant).

1.10.29 Prove that $\lambda(xB) = x\lambda(B)$ for every Borel set B and $x > 0$. Extend the result to all measurable sets.

1.10.30 Prove that for an measurable set of a finite measure M and n $\varepsilon > 0$ there is a set of the form $I = \bigcup_{i \leqslant n}(a_i, b_i)$ such that $\lambda(M \bigtriangleup I) < \varepsilon$ and $a_i, b_i \in \mathbb{Q}$.

Properties of measures

1.10.31 Let (X, Σ, μ) be a finite measure space. Prove that if $A_n \in \Sigma$ and for every n we have $\mu(A_n) \geqslant \delta > 0$ then there is $x \in X$ such that $x \in A_n$ for infinitely many n.

1.10.32 Prove that if (A_n) is a sequence of sets from a σ-algebra on which a finite measure μ is defined then the convergence of (A_n) to A implies $\mu(A) = \lim_n \mu(A_n)$. Is the assumption of the finiteness of the measure essential?

1.10.33 Let (X, Σ, μ) be a measure space. A set $T \in \Sigma$ is called an atom of μ if $\mu(T) > 0$ and for every $A \in \Sigma$ with $A \subseteq T$ either $\mu(A) = 0$ or $\mu(A) = \mu(T)$. We say that μ is **nonatomic** if it has no atoms.

Check that the Lebesgue measure is nonatomic. Note that our first examples of measures were all atomic.

1.10.34 Prove that a finite nonatomic measure μ on Σ has the following Darboux property: for any $A \in \Sigma$ and $0 \leqslant r \leqslant \mu(A)$ there is $B \in \Sigma$ such that $B \subseteq A$ and $\mu(B) = r$.

HINT: Suppose that $\mu(X) = 1$; check that if $\varepsilon > 0$ and $A \in \Sigma$ satisfies $\mu(A) > 0$ then there is $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \varepsilon$. Then prove that X is a uinion of pairwise disjoint A_n satisfying $0 < \mu(A_n) < \varepsilon$. This will show the the set of values of μ is dense in $[0, 1]$, and we are almost done.

Ideals and outer measures

1.10.35 A nonempty family $\mathcal{J} \subseteq \mathcal{P}(X)$ is a σ–ideal if $A \subseteq B$ and $B \in \mathcal{J}$ imply $A \in \mathcal{J}$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}$ whenever $A_n \in \mathcal{J}$ for $n = 1, 2, \ldots$. Think of σ–ideals on $\mathbb{R}$ i $\mathbb{R}^2$ you know.

1.10.36 Let $\mathcal{J}$ be a σ–ideal on X. Describe $\mathcal{A} = \sigma(\mathcal{J})$ (consider two cases: $X \in \mathcal{J}, X \notin \mathcal{J}$). Define a 0-1 measure μ on $\mathcal{A}$ in a natural way (compare section 1.2).

1.10.37 Let $\mathcal{J} \subseteq \mathcal{P}(X)$ be a σ–ideal not containing X. On $a(\mathcal{J})$ we consider μ as above. Consider the outer measure induced by μ and characterize the family of measurable sets.

1.10.38 Consider a partition of a space X into nonempty sets $\{A_1, A_2, \ldots\}$.

(i) Describe an algebra $\mathcal{A}$ generated by all A_n, $n \in \mathbb{N}$.

(ii) Define an additive set-function μ on $\mathcal{A}$ so that $\mu(A_n) = 2^{-n}$ and $\mu(X) = 1$. How one can describe sets that are measurable with respect to μ^*? (see Definition 1.9.1)

1.10.39 Consider $X = [0, 1) \times [0, 1]$ and let $\mathcal{R}$ be an algebra in X generated by all the cylinders $[a, b) \times [0, 1]$. Consider μ on $\mathcal{R}$ such that $\mu([a, b) \times [0, 1]) = b - a$ for $0 \leqslant a < b \leqslant 1$. Get an idea of μ^*–measurable sets (see Definition 1.9.1). Note that one can define *many* pairwise disjoint nonmeasurable sets satisfying $\mu^*(E) = 1$.

1.10.40 Let $\mathcal{R}$ be a ring of subsets of $\mathbb{Q}$ generated by all the sets $\mathbb{Q} \cap [a, b)$ $(a, b \in \mathbb{R})$. Check that one can define an additive function ν on $\mathcal{R}$ such that $\nu(\mathbb{Q} \cap [a, b)) = b - a$ for $a < b$. Prove that ν is not countable additive on $\mathcal{R}$ and calculate $\nu^*(\mathbb{Q})$.

1.10.41 Note that in the formula for λ^* we can replace intervals $[a, b)$ by those of the form (a, b) (or of the form $[a, b]$). This shows directly that Lebesgue measurable sets can be approximated from above by open sets.

1.11 Problems

1.11.A Prove that any union (even uncountable) of intervals of the form $[a, b]$, where $a < b$, is a Borel set.

1.11.B Prove that for any space X, $|X| \leqslant \mathfrak{c}$ if and only if there is a countable family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $\sigma(\mathcal{F})$ contains of the singletons.

1.11.C Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of size $\leqslant \mathfrak{c}$. Prove that $|\sigma(\mathcal{F})| \leqslant \mathfrak{c}$. Conclude that $|Bor(\mathbb{R})| = \mathfrak{c}$ so there are non-Borel subsets of $\mathbb{R}$..

Remark: here transfinite induction is needed.

1.11.D Prove that λ defined on the ring generated by all the intervals $[a, b)$ (by $\lambda([a, b)) = b - a$ for $a < b$) continuous from above on $\emptyset$ (so is countably additive). Hint: Sets like $\bigcup_{i=1}^{n} [c_i, d_i]$ are compact and (in a sense) approximate sets from $\mathcal{R}$ from below.

1.11.E Let (X, Σ, μ) be a probability measure space and let $A_1, \ldots, A_{2009} \in \Sigma$ satisfy $\mu(A_i) \geqslant 1/2$. Prove that there is $x \in X$ such that $x \in A_i$ for at least 1005 many i.

1.11.F Carry out the classical Vitali construction: For $x, y \in [0, 1)$, $x \sim y \iff x - y \in \mathbb{Q}$. Check that $\sim$ is an equivalence relation. Let Z be a set choosing exactly one element from each equivalence classes Check that $\bigcup_{q \in \mathbf{Q}} (Z \oplus q) = [0, 1)$, where $\oplus$ denotes addition mod 1.

Note that λ is invariant on $[0, 1)$ with respect to $\oplus$; prove that Z is not Lebesgue measurable.

1.11.G Construct a Borel set $B \subseteq \mathbb{R}$ such that $\lambda(B \cap I) > 0$ and $\lambda(B^c \cap I) > 0$ for every nonempty open interval I.

1.11.H Prove Steinhaus' theorem: If $A \subseteq \mathbb{R}$ is measurable and $\lambda(A) > 0$ then $A - A$ (algebraic difference) contains $(-\delta, \delta)$ for some $\delta > 0$.

Hint: We can assume that $\lambda(A) < \infty$; show first that there is a nonempty open interval I such that $\lambda(A \cap I) \geqslant \frac{3}{4} \lambda(I)$.

1.11.I Let $A \subseteq \mathbb{R}$ be such a set that $\lambda(A \bigtriangleup (x + A)) = 0$ for every rational number x. Prove that either $\lambda(A) = 0$ or $\lambda(\mathbb{R} \setminus A) = 0$.

Hint : Steinhaus' theorem.

1.11.J (Requires transfinite induction.) Construct the so call Bernstein set $Z \subseteq [0, 1]$, i.e. such a set that

$$Z \cap P \neq \emptyset, \quad P \setminus Z \neq \emptyset,$$

for every uncountable closed $P \subseteq [0, 1]$. Then note that Z is not Lebesgue measurable, in fact $\lambda^*(Z) = \lambda^*([0, 1] \setminus Z) = 1$.

Hint: Enumerate all such P as P_α, $\alpha < \mathfrak{c}$. Define Z to be $\{z_\alpha : \alpha < \mathfrak{c}\}$, where z_α and y_α satisfy

$$z_\alpha, y_\alpha \in P_\alpha \setminus \{z_\beta, y_\beta : \beta < \alpha\}.$$

To make it work you have to know or prove it that every P_α has cardinality $\mathfrak{c}$.