4.6 Exercises

4.6.1 Let $f : \mathbb{R} \to \mathbb{R}_+$ be a Borel function. Check that $\{(x, y) : 0 \leqslant y \leqslant f(x)\}$ is a Borel subset of the plane.

4.6.2 Let $f : X \to \mathbb{R}_+$ be a nonnegative measurable function on the space (X, Σ, μ); consider $P = \{(x, t) : 0 \leqslant t \leqslant f(x)\}$, the set below the graph of f. Check that P belongs to the σ-algebra $\Sigma \otimes Bor(\mathbb{R})$ and conclude from the Fubini theorem that

$$\mu \otimes \lambda(P) = \int_X f \, d\mu.$$

4.6.3 Note that a Borel set $A \subseteq [0, 1]^2$ is of planar Lebesgue zero if and only if $\lambda(A_x) = 0$ for almost all $x \in [0, 1]$.

4.6.4 Note that if Borel sets $A, B \subseteq [0, 1]^2$ satisfy $\lambda(A_x) = \lambda(B_x)$ for all x then $\lambda_2(A) = \lambda_2(B)$.

4.6.5 Calculate the Lebesgue measure of those two sets:

$$A = \{(x, y) : \ x \in \mathbb{Q} \text{ lub } y \in \mathbb{Q}\}; \qquad B = \{(x, y) : \ x - y \in \mathbb{Q}\}.$$

4.6.6 Using a well-known fact that that isometries of the plane do not change the area of rectangles, prove that the planar Lebesgue measure is invariant with respect to all isometries.

4.6.7 Prove that the planar Lebesgue measure satisfies the formula $\lambda_2(J_r[B]) = r^2\lambda_2(B)$ dla $B \in Bor(\mathbb{R}^2)$, where J_r is a homothety of ratio r.

4.6.8 Derive from the Fubini theorem

(i) a formula for the volume of a cone of height h whose base is a Borel set $B \subseteq \mathbb{R}^2$;

(ii) a formula for the volume of the ball of radius r in $\mathbb{R}^3$ i $\mathbb{R}^4$.

4.6.9 Note that the measure $\lambda \otimes \lambda$ is not complete on $\mathfrak{L} \otimes \mathfrak{L}$.

4.6.10 Let ν be the counting measure on the family of all subsets of $\mathbb{N}$. Give an example of a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ for which the iterated integrals in the Fubini formula give different finite values.

HINT: Define some nonzero values of $f(n, n)$ i $f(n + 1, n)$ for all $n \in \mathbb{N}$.

4.6.11 Consider the following two functions on the unit square

$$f(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad g(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},$$

$f(0, 0) = g(0, 0) = 0$. Check if those functions are integrable and if the iterated integrals exist and are equal; compare the observations with the Fubini theorem.

4.6.12 Prove that an integrable function $f : [0,1]^2 \to \mathbb{R}$ satisfies

$$\int_0^1 \int_0^x f(x,y)\, \mathrm{d}\lambda(y)\, \mathrm{d}\lambda(x) = \int_0^1 \int_y^1 f(x,y)\, \mathrm{d}\lambda(x)\, \mathrm{d}\lambda(y).$$

4.6.13 Let $\mathcal{A}$ be a σ–algebra of subsets of $[0,1]$ generated by countable sets. Prove that the diagonal $\Delta = \{(x,y) \in [0,1]^2 : \ x = y\}$ is not in $\mathcal{A} \otimes \mathcal{A}$.

4.6.14 A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be Borel if $f^{-1}[B] \in Bor(\mathbb{R}^n)$ for $B \in Bor(\mathbb{R}^k)$. Here $Bor(\mathbb{R}^n)$ denotes the σ-algebra generated by all open subsets of $\mathbb{R}^n$. Check that

(i) $Bor(\mathbb{R}^2)$ is generated by open rectangles $U \times V$;

(ii) $Bor(\mathbb{R}^n)$ is generated by open serts of the form $U_1 \times U_2 \times \ldots \times U_n$;

(iii) every continuous $f : \mathbb{R}^n \to \mathbb{R}$ is Borel;

(iv) a function $g = (g_1, g_2) : \mathbb{R} \to \mathbb{R}^2$ is Borel if and only if the functions g_1, g_2 are Borel.

4.6.15 Conclude from the above that if $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are measurable functions then the functions $g_1 + g_2, g_1 \cdot g_2$ are also measurable.

4.6.16 Let $f : X \to Y$ be a measurable mapping between (X, Σ, μ) and $(Y, \mathcal{A})$, that is we have $f^{-1}[A] \in \Sigma$ for every $A \in \mathcal{A}$. Check that the formula $\nu(A) = \mu(f^{-1}[A])$ defines a measure on $\mathcal{A}$. Such a measure is called the image measure (of μ under f); we denote it as $\nu = f[\mu]$.

4.7 Problems

4.7.A Assuming the continuum hypothesis one can order the interval by the relation $\prec$ so that every initial segment $\{a : a \prec b\}$ is this order is countable for every $b \in [0,1]$. Note that then the set

$$Z = \{(x,y) \in [0,1] \times [0,1] : x \prec y\},$$

does not satisfy the Fubini theorem and hence it is not measurable on the plane.

4.7.B Find a set A on the plane of planar Lebesgue zero and such that A meets every rectangle of positive measure.

Hint : First, generalize Steinhaus' theorem to the following: if $A, B \subseteq \mathbb{R}$ have positive measure then $A - B$ contains a rational number.

4.7.C Let $\Delta = \{(x,x) : x \in X\}$ be the diagonal. Prove that Δ belongs to $\mathcal{P}(X) \otimes \mathcal{P}(X)$ if and only if $|X| \leqslant \mathfrak{c}$.

4.7.D Let

$$h : \{0,1\}^{\mathbb{N}} \to [0,1], \quad h(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n}.$$

Check that h is a continuous function so it is measurable with respect to σ–algebra $Bor\{0,1\}^{\mathbb{N}}$; moreover, $h[\{0,1\}^{\mathbb{N}}] = [0,1]$.

Prove that λ on $[0,1]$ is the image of the Haar measure ν on $\{0,1\}^{\mathbb{N}}$ by this function.

4.7.E Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be the set of those x that contain, at least once, a fixed finite sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of zeros and ones. Prove that $\nu(A) = 1$.

4.7.F Prove that $\nu(x \oplus A) = \nu(A)$ for every Borel subset A of the Cantor set.

Hint: Check first the formula for sets C from the algebra $\mathcal{C}$ defined in 4.5.

4.7.G A Borel set $A \subseteq \{0,1\}$ is called a *tail set* if $e \oplus A = A$ for every $e \in \{0,1\}$ for which $e(n) = 0$ for almost all n. Prove that $\nu(A) = 0$ or $\nu(A) = 1$ for every tail set A (this is so called Kolmogorov's 0-1 law).

Hint : If A is a tail set then $\nu(A \cap C) = \nu(A)\nu(C)$ for every $C \in \mathcal{C}$; use the fact that $\nu(A \triangle C)$ can be arbitrarily small.

4.7.H Let X be a finite set and let μ be a measure defined for all subset of $X \times X$, vanishing on the diagonal. Prove that there are disjoint $A, B \subseteq X$ such that $\mu(A \times B) \geqslant 1/4$.