5.5 Exercises

5.5.1 Note that the Hahn decomposition $X = X^+ \cup X^-$ for a signed measure κ is unique "up to sets of measure zero" (what does it mean?). Check if the decomposition of a signed measure into a difference of two measures is also unique.

5.5.2 Note that if a signed measure ν takes only real values then it is bounded.

5.5.3 Let f be a measurable function such that at least one of functions f^+, f^- is μ–integrable. let $\nu(A) = \int_A f \, \mathrm{d}\mu$ for $A \in \Sigma$ (here μ is a measure on Σ). Write ν^+, ν^- and $|\nu|$ using some integrals.

5.5.4 Note that for a signed measure ν, $|\nu|(A) = 0$ if and only if $\nu(B) = 0$ for every $B \subseteq A$ $(A, B \in \Sigma)$.

5.5.5 Observe that if $\nu \ll \mu$ and $\nu \perp \mu$ then $\nu = 0$.

5.5.6 Note that $\nu \ll \mu$ if and only if $\nu^+, \nu^- \ll \mu$; an analogous property holds for singularity of measures.

5.5.7 RN theorem need not hold for measures μ that are not σ–finite. Let Σ be a σ–algebra generated by all countable subsets of $[0, 1]$; consider the counting measure μ on Σ and a 0-1 measure ν on Σ.

5.5.8 Complete the details of the proof of Corollary 5.3.2 following the sketch given there.

5.5.9 Let μ, ν be σ–finite measures on Σ such that $\nu \ll \mu$ and $\mu \ll \nu$. prove that we have almost everywhere

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 1 / \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

5.5.10 Let μ, ν be σ–finite with $\nu \ll \mu$ and let the function $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ be positive everywhere. Check that $\mu \ll \nu$.

5.5.11 Let (X, Σ, μ) be a probability measure space and let $\mathcal{A}$ be a σ–algebra contained in Σ.

Prove that for every Σ–measurable integrable function $f : X \to \mathbb{R}$ there is an $\mathcal{A}$–measurable function g such that for every $A \in \mathcal{A}$

$$\int_A g \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu.$$

(In probability, such $g = E(f|\mathcal{A})$ is called the conditional expectation of f).

5.5.12 A distribution function of a probability measure μ on $Bor(\mathbb{R})$ is $F_\mu : \mathbb{R} \to \mathbb{R}$ given by the formula $F_\mu(x) = \mu(-\infty, x)$ for $x \in \mathbb{R}$. Check that such F_μ is nondecreasing and left-continuous; moreover, $\lim_{x\to\infty} F_\mu(x) = 1$.

Remark: One can also define $F_\mu(x) = \mu(-\infty, x]$; how does this change properties of F_μ?

5.5.13 Prove that a distribution function F_μ is continuous if and only if μ vanishes on points.

5.5.14 A measure vanishing on points is sometimes called continuous. Prove that a probability measure μ on $Bor(\mathbb{R})$ is continuous if and only if it is nonatomic.

5.5.15 As we already know (!), there is a continuous probability measure μ on the usual Cantor set C. Let $F(x) = \mu((-\infty, x))$ be the distribution function of such a measure. Check that F is continuous and $F[C] = [0, 1]$.

Conclude that an image of a set of measure zero by a continuous function need not be of measure zero and can be even nonmeasurable.

5.5.16 Calculate (or bring to a familiar form); explain calculations:

(i) $\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu$ where $\mu = \delta_0$, $\mu = \delta_0 + \delta_1$, $\mu = \sum_{n=1}^{\infty} \delta_n$ (here δ_x denotes the point mass at x).

(ii) $\int_{[0,1]} x^2\,\mathrm{d}\lambda$;

(iii) $\int_{[0,1]} f\,\mathrm{d}\lambda$; where $f(x) = x$ for $x \notin \mathbb{Q}$, $f(x) = 0$ for $x \in \mathbb{Q}$;

(iv) $\int_{[0,2\pi]} \sin x\,\mathrm{d}\mu$, where $\mu(A) = \int_A x^2\,\mathrm{d}\lambda(x)$;

(v) $\int_{\mathbb{R}} f\,\mathrm{d}\lambda$; where $f(x) = x^2$ for $x \in \mathbb{Q}$, $f(x) = 0$ for $x \notin \mathbb{Q}$;

(vi) $\int_{\mathbb{R}} 1/(x^2+1)\,\mathrm{d}\lambda(x)$;

(vii) $\int_{\mathbb{R}} \cos x\,\mathrm{d}\mu$, where $\mu(A) = \int_A 1/(x^2+1)\,\mathrm{d}\lambda(x)$;

(viii) $\int_{\mathbb{R}} \cos x\,\mathrm{d}\mu$, where μ satisifes $\mu(-\infty, x) = \arctan x + \pi/2$;

(ix) $\int_{[0,\infty)} [x]\,\mathrm{d}\mu$, where μ is such that $\mu[n, n+1) = n^{-3}$;

(x) $\int_{\mathbb{R}} (x - [x])\,\mathrm{d}\mu$, where

$$\mu = \sum_{n=1}^{\infty} \delta_{n+1/n};$$

(xi)

$$\lim_{n\to\infty} \int_{[0,1]} \frac{n^2x+2}{n^2x+n+3}\,\mathrm{d}\lambda(x) \qquad \lim_{n\to\infty} \int_{[0,\infty]} \frac{n}{xn^2+3}\,\mathrm{d}\lambda(x).$$

5.5.17 Let $f : X \to \mathbb{R}$ be a measurable function on a measure space (X, Σ, μ). Then the formula $\nu(B) = \mu(f^{-1}[B])$ defines a Borel measure on $\mathbb{R}$, cf. Exercise 16 from the previous chapter (in probability such a measure is called a distribution of a random variable)

Prove that $\int_X f\,\mathrm{d}\mu = \int_{\mathbb{R}} x\,\mathrm{d}\nu(x)$ (for f integrable).

Hint: Consider first $f = \chi_A$ for $A \in \Sigma$; then simple functions and so on.

5.6 Problems

5.6.A Let (X, Σ, μ) be a measure space. For any $Z \subseteq X$ we write $\mu^*(Z) = \inf\{\mu(A) : A \in \Sigma, Z \subseteq A\}$. Note that μ^* is an outer-measure (is countably subadditive and monotone) but need not be additive.

Prove that for a fixed $Z \subseteq X$, the formula $\nu(A \cap Z) = \mu^*(A \cap Z)$ defines a measure on the σ–algebra $\{A \cap Z : A \in \Sigma\}$ of subsets of Z.

5.6.B There is a space $Z \subseteq [0, 1]$ and a probability measure ν on $Bor(Z)$ such that $\nu(K) = 0$ for every compact $K \subseteq Z$.

Hint: Take first a nonmeasurable $Z \subseteq [0, 1]$ and consider the measure from the previous problem.

5.6.C Let (X, Σ, μ) be a probability space. As we know, $A \sim B \iff \mu(A \triangle B) = 0$ defines the equivalence relation Let $\mathfrak{B} = \{[A] : A \in \Sigma\}$ be the family of the equivalence classes.

Check that one can equip $\mathfrak{B}$ with the natural operations

$$[A] \vee [B] = [A \cup B], \quad [A] \wedge [B] = [A \cap B], \quad -[A] = [A^c].$$

Then $\mathfrak{B}$ becomes a Boolean algebra $(\mathfrak{B}, \vee, \wedge, -, 0, 1)$ (that is, those operation have properties analogous to the usual set-theoretic ones; $0 = [\emptyset]$, $1 = [X]$). Such a Boolean algebra is called the measure algebra.

5.6.D Check that the measure algebra $\mathfrak{B}$ is a metric space when we measure distances by the formula $d([A], [B]) = \mu(A \triangle B)$. Prove that the metric in question is complete.

5.6.E The measure algebra of the Lebesgue measure λ on $[0, 1]$ is a separable metric space.