6.6 Exercises

6.6.1 Check that $|a+b|^p \leqslant 2^{p/q}(|a|^p+|b|^p)$, where $1/p+1/q=1$; conclude that $L_p(\mu)$ is a linear space.

6.6.2 Check the the facts mentioned below can be proved following the argument for $L_1(\mu)$ ($p \geqslant 1$)

(i) $L_p(\mu)$ is complete;

(ii) simple functions are dense in $L_p(\mu)$;

(iii) $C[0,1]$ is dense in $L_p[0,1]$.

6.6.3 Check if there are any inclusions between $L_p(\mathbb{R})$ for various p. Consider the same question for $L_p[0,1]$.

6.6.4 Check whether the following statements are always true or if they hold under the assumption $\mu(X)<\infty$; here f_n is a sequence of measurable functions.

(i) if f_n are integrable and converge uniformly to f then f_n converge in L_1;

(ii) if f_n are integrable and converge to f almost uniformly then f_n converge in L_1;

(iii) if $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots$ and $\sup_n \int f_n \, \mathrm{d}\mu < \infty$ then the limit is integrable;

(iv) if f_n converge in $L_1(\mu)$ then some subsequence converges almost everywhere;

(v) if f_n are integrable and converge to 0 almost everywhere then f_n are equi-integrable;

(vi) if $|f_n| \leqslant g$ where $\int g \, \mathrm{d}\mu < \infty$ then f_n are equi-integrable;

(vii) if $|f_n| \leqslant g$, $\int g \, \mathrm{d}\mu < \infty$, f_n converge almost everywhere then f_n converge in $L_1(\mu)$

(viii) if $f_n \in L_2(\mu) \cap L_1(\mu)$ and f_n converge in $L_1(\mu)$ then f_n converge in $L_2(\mu)$; vice versa?

(ix) consider (viii) for uniformly bounded f_n.

6.6.5 Note that for a function $f: X \to \mathbb{C}$, $f = f_1 + i \cdot f_2$ its measurability is equivalent to measurability of both the real f_1 and imaginary part f_2. Moreover, f is integrable if and only if f_1, f_2 are integrable.

6.6.6 For a function $f: X \to \mathbb{R}$ on a given space (X, Σ, μ) we write $||f||_\infty$ for its essential supremum which is

$$||f||_\infty = \inf\{\sup_{X \setminus A} |f| : \mu(A) = 0\}.$$

Prove that $||\cdot||_\infty$ is a complete norm on the space $L_\infty(\mu)$ of those functions that satisfy $||f||_\infty < \infty$, when we identify functions equal almost everywhere.

6.6.7 Prove that for $f \in L_\infty[0,1]$ we have $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

6.6.8 Prove that the space $L_\infty[0,1]$ is not separable.

6.6.9 We say that a measure μ is separable if $L_1(\mu)$ is separable as a Banach space. Prove that μ is separable if and only if there is a countable family $\mathcal{S} \subseteq \Sigma$ such that for every $A \in \Sigma$

$$\inf\{\mu(A \triangle S) :\ S \in \mathcal{S}\} = 0.$$

6.7 Problems

6.7.A Consider a nonatomic probability measure space (X, Σ, μ). Prove that there is a measurable function $f : X \to [0,1]$ such that $f[\mu] = \lambda$.

HINT: It is enough to define $g : X \to \{0,1\}^{\mathbb{N}}$ with $g[\mu] = \nu$, where ν is the Haar measure on the Cantor set. For every n choose disjoint $A_\varepsilon \in \Sigma$, $\varepsilon \in \{0,1\}^n$ so that $\mu(A_\varepsilon) = 2^{-n}$ and $A_{\varepsilon\frown 0} \cup A_{\varepsilon\frown 1} = A_\varepsilon$.

6.7.B Prove that if (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) are nonatomic separable probability measure spaces then the corresponding measure algebras are isomorphic, that is there is a bijection between then preserving Boolean operations and their metric structures.

HINT: Pick $A_\varepsilon \in \Sigma_1$ as in Problem A with the property that the family $\mathcal{S}_1$ of finite unions of sets from A_ε, $\varepsilon \in \{0,1\}^n$, $n \in \mathbb{N}$ is dense. Choose $B_\varepsilon \in \Sigma_2$ analogously.

Define $g([A_\varepsilon]) = [B_\varepsilon]$ and extend g to $\mathcal{S}_1$ preserving Boolean operations; then g is an isometry so it can be extended to the closure.

6.7.C Prove that for the measures as above, $L_p(\mu_1)$ is linearly isometric to $L_p(\mu_2)$ (here $1 \leqslant p \leqslant \infty$).

HINT: Define a linear mapping $T : L_p(\mu_1) \to L_p(\mu_2)$, first on simple functions. Use the fact that every isometry defined on a subset of a metric space can be extended to the closure of its domain.

6.7.D (if you are familiar with ultrafilters). Let $\mathcal{F}$ be a non-principial ultrafilter on $\mathbb{N}$. Prove that the set $Z \subseteq \{0,1\}^{\mathbb{N}}$, where

$$Z = \{\chi_F :\ F \in \mathcal{F}\},$$

is not measurable with respect to the Haar measure.

HINT: Such a set is a tail set so, if measurable, it has either measure 0 or 1. consider the translation of Z by the constant 1 element of the group.

6.7.E How many measures (finite, σ-finite, arbitrary) one can define on $Bor(\mathbb{R})$?