16. Baire and Borel sets

We are not going to give a systematic account of various aspects of general topological measure theory. We shall mainly discuss measures on spaces such as $[0,1]^T$ or $\mathbb{R}^T$ that appear naturally in various parts of mathematics, for instance if you want to realize your favourite stochastic process.

The first problem that arises when we start considering nonmetrizable topological spaces is about the basic σ-algebra. In a metrizable space X, the Borel sets $Bor(X)$ is the smallest σ-algebra with respect to which all the continuous functions are measurable. Consider the following example.

Example 16.1. Let S be any uncountable set; take $X = S \cup \{\infty\}$ (∞ is some element outside S). We topologize X by declaring that every $\{s\}$ is open for $s \in S$, while open neighbourhoods of ∞ are of the form $V_I = (S \setminus I) \cup \{\infty\}$, where $I \subseteq S$ is finite. This is the one-point compactification of the discrete space S; X is a compact Hausdorff space.

Take any continuous function $f : X \to \mathbb{R}$; then for every n there is a finite set I_n such that $|f(x) = f(\infty)| < 1/n$ for $x \in V_{I_n}$. This implies that $f(x) = f(\infty)$ for every $x \in S \setminus \bigcup_n I_n$.

It follows that every continuous function on X is measurable with respect to the σ-algebra Σ of sets $E \subseteq X$, such that either $E \subseteq S$ is countable or $\infty \in E$ and $S \setminus E$ is countable. On the other hand, $Bor(X)$ is the power set of X.

Definition 16.2. In a topological space X, the Baire σ-algebra $Baire(X)$ is the smallest one with respect to which all continuous functions are measurable.

Of course, $Baire(X)$ is generated by all the sets of the form $f^{-1}[F]$, where $F \subseteq \mathbb{R}$ is closed and $f \in C(X)$. Since for a closed set $F \subseteq \mathbb{R}$ there is a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that $F = \theta^{-1}(0)$ we have $f^{-1}[F] = (\theta \circ f)^{-1}(0)$. Hence, a subset of X that is a preimage of a closed set by a continuous function is a preimage of $\{0\}$ by another continuous function. For that reason, such sets are called **zero sets** (recall that a set of measure zero is rather called a null set).

If Z is a zero set in X then Z is closed and G_δ: for a continuous $g : X \to \mathbb{R}$ with $Z = g^{-1}(0)$ we have $Z = \bigcap_n g^{-1}[(-1/n, 1/n)]$. This sometimes reverses.

Lemma 16.3. *In a normal space X every closed G_δ set Z is a zero set.*

Proof. We have $Z = \bigcap_n V_n$ for some open $V_n \subseteq X$. Define g_n on $Z \cup (X \setminus V_n)$ putting $g_n(x) = 0$ for $x \in Z$ and $g_n(x) = 1$ for $x \notin V_n$. Then g_n is continuous and extends to a continuous function $f_n : X \to [0,1]$, by the Tietze extension theorem (here we need normality of X).

Take now $f = \sum_n 2^{-n} f_n$; f is continuous (as the series converges uniformly), and we check that $f(x) = 0$ if and only if $x \in Z$. $\square$

Corollary 16.4. *In a normal topological space X, $Baire(X)$ is the smallest σ-algebra containing all closed G_δ sets.*

Recall that if $X = \prod_n X_n$ is a product of separable metrizable spaces then $Bor(X) = \bigotimes_n Bor(X_n)$. We shall see that the situation changes dramatically if we consider uncountable products.

Let $X = \prod_{t \in T} X_t$ be an arbitrary (possibly uncountable) product. Recall that we write $A \sim I$ for $A \subseteq X$ and $I \subseteq T$ to say that A is determined by coordinates in I, i.e. $A = \pi_I^{-1}[\pi_I[A]]$, where $\pi_I : X \to \prod_{t \in I} X_t$ is the projection. In other words, $A \sim I$ means that if we take $x \in A$ and change its coordinates outside I then it still belongs to A.

In the topological setting, every basic open set in $X = \prod_{t \in T} X_t$ is of the form

$$U = \bigcap_{i=1}^{n} \pi_{t_i}^{-1}[V_i] = \{x \in X : x_{t_i} \in V_i \text{ for all } i \le n\},$$

for some $t_i \in T$ and open sets $V_i \in X_{t_i}$, so every basic open set is determined by a finite number of coordinates.

Lemma 16.5. *If $X = \prod_{t \in T} X_t$ and every X_i is equipped with a σ-algebra Σ_t then every set A from the product σ-algebra $\Sigma = \bigotimes_t \Sigma_t$ is determined by countably many coordinates.*

Proof. Recall that, by definition, Σ is generated by all the sets $\pi_t^{-1}[B]$, $t \in T$, $B \in \Sigma_t$. Each such a generator is determined by one coordinate. It is enough to notice that the family of subsets of X determined by countably many coordinates is a σ-algebra. $\square$

Example 16.6. Take $K = [0,1]^T$ for uncountable T. Note that for no $x \in K$, the singleton $\{x\}$ is determined by countably many coordinates (it is actually determined by them all). Hence $\{x\} \notin \bigotimes_t Bor[0,1]$. In particular, the product σ-algebra is much smaller then $Bor(K)$.

The following explains the necessity of introducing Baire sets.

Theorem 16.7. *If $X = \prod_{t \in T} X_t$ is the product of separable metrizable spaces then*

$$\bigotimes_{t \in T} Bor(X_t) = Baire(X).$$

The harder part of Theorem 16.6 is to prove that every continuous function $f : X \to \mathbb{R}$ is measurable with respect to the product σ-algebra. This will be a consequence of the following general fact.

Theorem 16.8. *If $X = \prod_{t \in T} X_t$ is the product of separable metrizable spaces then for every continuous $f : X \to \mathbb{R}$ there is a countable set $I \subseteq T$ such that $f = f' \circ \pi_I$ for some continuous function $f' : \prod_{t \in I} X_t \to \mathbb{R}$.*

The philosophical conclusion is: *You can use only countable many variables to define a continuous function on a product space.* Note that we can say that the function f as above is determined by coordinates in I ($f(x) = f(y)$ whenever x and y have the same coordinates in I). There is a short proof of 16.8 in the compact case (see the next problem list); for a non-compact case we need some preparations.

For the rest of this section we fix a family $\{X_t : t \in T\}$ of separable metrizable spaces. We need to know that arbitrary such products are *ccc*; this is not suprising if $|T| \leq \mathfrak{c}$ because the product of $\mathfrak{c}$-many separable spaces is separable. There is a combinatorial proof of 16.9, it will be discussed later.

Lemma 16.9. *The product space $X = \prod_{t\in T} X_t$ is ccc, i.e. every pairwise disjoint family of nonempty open subsets of X is countable.*

Proof. For every t fix a probability measure $\mu_t \in P(X_t)$ which is positive on every nonempty open subset of X_t (for instance, take $\mu_t = \sum_n 2^{-n}\delta_{z_n}$ where $\{z_n : n \in \mathbb{N}\}$ is dense in X_t). Write μ for the product measure $\bigotimes_t \mu_t$; we know that μ is defined on the product σ-algebra $\Sigma = \bigotimes_t Bor(X_t)$.

Take any family $\mathcal{U}$ of nonempty open subsets of X. Note that every $U \in \mathcal{U}$ contains some nonempty basic open set V_U and $V_U \in \Sigma$. Then $\mu(V_U) > 0$ and this implies that $\{V_U : U \in \mathcal{U}\}$ is countable. Hence, $\mathcal{U}$ must be countable as well. □

Lemma 16.10. *If U_1, U_2 are disjoin open subsets of $X = \prod_{t\in T} X_t$ then there is $A \subseteq X$ depending on countably many coordinates such that $U_1 \subseteq A$ and $A \cap U_2 = \emptyset$.*

Proof. Consider the maximal pairwise disjoint family $\mathcal{V}$ of basic open sets contained in U_1 and consider $A = \overline{\bigcup \mathcal{V}}$. Then $U_1 \subseteq A$ by maximality of $\mathcal{V}$; clearly $A \cap U_2 = \emptyset$.

We know from Lemma 16.9 that $\mathcal{V}$ is countable; every $V \in \mathcal{V}$ depends on finitely many coordinates (since V is basic open set); hence, $\bigcup \mathcal{V}$ depends on countably many coordinates and so does the closure (an exercise). □

Proof. (of Theorem 16.8) Consider a continuous function $f : X \to \mathbb{R}$. Enumerate by (P_n, Q_n) the family of all pairs of disjoint open intervals in $\mathbb{R}$. For every n, $f^{-1}[P_n]$ and $f^{-1}[Q_n]$ are disjoint open sets so by Lemma 16.10 they can be separated by a set A_n depending on a countable set $I_n \subseteq T$. Put $I = \bigcup_n I_n$.

We claim that if $x, y \in X$, $x|I = y|I$ then $f(x) = f(y)$. Otherwise, if $f(x) \neq f(y)$ then there is n such that $f(x) \in P_n$ and $f(y) \in Q_n$ so $x \in A_n$ and $y \notin A_n$. But x and y agree on I, so they are the same on I_n, a contradiction with $A_n \sim I_n$.

We can fix $z \in \prod_{t\in T\setminus I}$ and set $f'(x') = f(x', z)$ for $x' \in \prod_{t\in I} X_t$; here (x', z) denotes the element of X that has coordinates of x' on I and those of z, otherwise. This correctly defines the required decomposition $f = f' \circ \pi_I$. □

Proof. (of Theorem 16.6) The inclusion $\bigotimes_{t\in T} Bor(X_t) \subseteq Baire(X)$ should be clear.

For the reverse inclusion use Theorem 16.8: for a continuous function $f : X \to \mathbb{R}$ we have $f = f' \circ \pi_I$ for some countable $I \subseteq T$. Here f' is a continuous function on a separable metrizable space $\prod_{t\in I} X_t$ so it is Borel on its domain. Recall that $Bor(\prod_{t\in I} X_t) = \bigotimes_{t\in I} Bor(X_t)$. This implies that f is measurable with respect to $\bigotimes_{t\in T} Bor(X_t)$. □

17. Baire or Borel measures?

People sometimes write, even in books, *take the usual product measure μ on* $[0,1]^T$ when they mean to say that we take the Lebesgue measure λ on $[0,1]$ and multiply it by itself $|T|$-many times. The previous section explains that for uncountable T such a measure μ is not Borel if defined only on the product σ-algebra; it is a *Baire measure* on the product space. So μ is defined on a rather small family of sets and that family contains no singletons. We may feel insecure with that but there is a remedy. It will turn out that $Bor([0,1]^T)$ is contained in the completion of $Baire([0,1]^T)$ with respect to μ (the completion is obtained by adding subsets of sets of measure zero).

The situation is worse if we consider the same product measure μ on $(0,1)^T$: for countable T such a product is a Polish space so every measure on it is tight. For uncountable T the measure μ is not tight, even if we think that we can define it on all Borel sets.

Indeed, take any compact set $K \subseteq (0,1)^T$. Then for every $t \in T$, $\pi_t[K]$ is a compact subset of $(0,1)$ and therefore $\lambda(\pi_t[K]) < 1$. If T is uncountable then there is $\delta > 0$ such that $\lambda(\pi_t[K]) < 1-\delta$ for t from some uncountable $T_0 \subseteq T$. If we take a sequence of distinct $t_n \in T_0$ then we conclude that

$$\mu(K) \leq \mu\left(\bigcap_{k\leq n} \pi_{t_k}^{-1}[\pi_{t_k}[K]]\right) \leq (1-\delta)^n.$$

for every n. Hence, $\mu(K) = 0$.

Discussion on that drama coming soon!