18. Measures on uncountable products

The status of 'the usual product measure' on $[0,1]^T$ is clarified by Kakutani's theorem below. The main conclusion here, given in the final section, is that, though generally, the Baire σ-algebra $Baire(X)$ in a topological space X is much smaller that the Borel one, 'nice' Baire measures extends uniquely to 'nice' Borel measures. This, in particular, applies to product measures on arbitrary products of separable metrizable spaces.

Recall first the concept of completion of a measure space. If (X, Σ, μ) is any measure space then by $(\Sigma)_\mu$ we denote the completion of Σ with respect to μ, that is $(\Sigma)_\mu$ is generated by Σ and all the subsets of sets from Σ of measure zero. Every $E \in (\Sigma)_\mu$ can be written as $E = A \triangle N$, where $A \in \Sigma$, $N \subseteq B \in \Sigma$, $\mu(B) = 0$. The measure μ can be extended to that completion, simply by $\widetilde{\mu}(A \triangle N) = \mu(A)$.

Definition 18.1. A subset V of a topological space X is called a *cozero set* if $X \setminus V$ is a zero set. Equivalently, $V = \{x \in X : g(x) \neq 0\}$ for some continuous function $g : X \to \mathbb{R}$.

This is an exercise (on the problem list) to check that the family of cozero sets is closed under finite intersections and countable unions.

Theorem 18.2 (Kakutani)**.** *Let $\{K_t : t \in T\}$ be any family of compact metrizable spaces and let $\mu_t \in P(K_t)$ be a strictly positive probability measure on K_t for every $t \in T$. Write $K = \prod_{t \in T} K_t$.*

If μ is the product measure defined on $Baire(K) = \bigotimes_t Borel(K_t)$ then

$$Bor(K) \subseteq (Baire(K))_\mu,$$

so μ becomes a Borel measure when considered on the completion of its domain.

Proof. Note that it is enough to prove that $U \in (Baire(K))_\mu$ for every open set $U \subseteq K$. For such a set U we shall find two cozero sets V_1, V_2 (here open sets depending on countably many coordinates) such that $V_1 \subseteq U \subseteq V_2$ and $\mu(V_2 \setminus V_1) = 0$. Then $V_1, V_2 \in Baire(K)$ so this will imply that U lies in the completion of the Baire σ-algebra.

Let r be the supremum of the values of $\mu(V)$ where $V \subseteq U$ is a cozero set. It is easy to check that the supremum is attained so there is a cozero set $V_1 \subseteq U$ with $\mu(V_1) = r$. Say that V_1 depends on coordinates in a countable set $I \subseteq T$. We define

$$V_2 = \pi_I^{-1}\left[\pi_I[U]\right].$$

Then $V_1 \subseteq U \subseteq V_2$, V_2 is an open set depending on I so it remains to check that $\mu(V_2 \setminus V_1) = 0$.

Suppose that $\mu(V_2 \setminus V_1) > 0$. Write $V_1' = \pi_i[V_1]$ and $V_2' = \pi_I[V_2]$. Then V_1', V_2' are open subets of the countable product $K' = \prod_{t \in I} K_t$, and if we write μ' for the corresponding product measure $\prod_{t \in I} \mu_t$ then $\mu'(V_2' \setminus V_1') = \mu(V_2 \setminus V_1) > 0$.

By the properties of measures on metrizable spaces, such as K', there is a closed set $F \subseteq V_2' \setminus V_1'$ such that $\mu'(F) > 0$ and $\mu'(F \cap W) = 0$ implies $W \cap F = \emptyset$ for every open

$W \subseteq K'$ (just remove from F all relatively open sets of measure zero). Take now any $x' \in F$; then there is $x'' \in \prod_{t\in T\setminus I} K_t$ such that $x = (x', x'') \in U$ (by the definition of V_2). By openess of U, there is an open basic set $G \subseteq K$ such that $x \in G \subseteq U$. Using our prime-and-double prime convention we can write $G = G' \times G''$ where $G' \subseteq K'$. We have $\mu(G) > 0$ but also

$$\mu(G \setminus V_1) \geq \mu((F \cap G') \times G'')) = \mu'(F \cap G) \cdot \mu''(G'') > 0,$$

and this is a contradiction with maximality of the value of r defined above. □

In particular, the usual product measure on $[0,1]^T$ is a Borel measure when completed. We have noted that the product measure on $(0,1)^T$ need not be tight. However, it must have some good property as the underlying space looks canonical; it is singled out below.

Definition 18.3. A probability measure on $Baire(X)$ is said to be τ-additive if for every cover $\mathcal{V}$ of X by cozero sets we have $\mu(\bigcup \mathcal{V}_0) = 1$ for some countable $\mathcal{V}_0 \subseteq \mathcal{V}$.

This definition may be rephrased as follows. Say that $\mathcal{V}$ us directed if for every $V_1, V_2 \in \mathcal{V}$ there is $V \in \mathcal{V}$ such that $V_1 \cup V_2 \subseteq V$. The τ-additivity of μ is equivalent to saying that for every directed family $\mathcal{V}$ of cozero sets, if $\bigcup \mathcal{V} = X$ then $\sup_{V\in\mathcal{V}} \mu(V) = 1$.

Theorem 18.4. *Let $\{X_t : t \in T\}$ be any family of metrizable separable spaces and let $\mu_t \in P(K_t)$ be a probability measure on X_t for every $t \in T$. Write $X = \prod_{t\in T} X_t$.*

If μ is the product measure defined on $Baire(X) = \bigotimes_t Borel(X_t)$ then μ is τ-additive.

Proof. The argument is similar to that from 18.2 (so the proof will be sketchy). Take a directed cover $\mathcal{V}$ of X by cozero sets and consider $r = \sup_{V\in\mathcal{V}} \mu(V)$. Note that there is $W = \bigcup_n V_n$ for some $V_n \in \mathcal{V}$ such that $\mu(V_n) \to r = \mu(W)$; moreover W depends on countably many coordinates $I \subseteq T$.

Decompose everything to parts from $X' = \prod_{t\in I} X_t$ and $X'' = \prod_{t\in T\setminus I} X_t$: $W = W' \times W''$, $\mu(W) = \mu'(W') \cdot 1 = r$. Then define ν on $X' \setminus W'$ by $\nu(B) = \mu(B \times X'')$ to conclude that for every $x' \in X'$, ν vanishes on some neighbourhood of x' so $\nu = 0$, that is $r = 1$. □

19. Regular extensions of finitely additive measures

We need some tool to extend Baire measures to Borel ones. We consider here an algebra $\mathcal{A}$ of subsets of some space X (no topology is involved at the beginning). Write simply $P(\mathcal{A})$ for the collections of all finitely additive probabilities on $\mathcal{A}$.

We shall call a family $\mathcal{L}$ of subsets of X a **lattice** if $\emptyset, X \in \mathcal{L}$ and $\mathcal{L}$ is closed under finite unions and intersections.

Definition 19.1. Given $\mu \in P(\mathcal{A})$ and a lattice $\mathcal{L} \subseteq \mathcal{A}$, we say that μ is $\mathcal{L}$-regular if $\mu(A) = \sup\{\mu(L) : L \in \mathcal{L}, L \subseteq A\}$ for every $A \in \mathcal{A}$.

We have seen examples of regularity: every σ-additive measure on a separable metrizable space X is regular with respect to the lattice of closed subsets of X.

Theorem 19.2. *Let $\mathcal{A}$ be an algebra of subsets of X, $\mathcal{K} \subseteq \mathcal{A}$ be a lattice contained in another lattice $\mathcal{L}$.*

Then every $\mathcal{K}$-regular $\mu \in P(\mathcal{A})$ can be extended to an $\mathcal{L}$-regular finitely additive measure on the algebra generated by $\mathcal{A} \cup \mathcal{L}$.

Proof. As in L7/P7 we write

$$\mu^*(Y) = \inf\{\mu(A) : \ Y \subseteq A \in \mathcal{A}\}, \quad \mu_*(Y) = \sup\{\mu(A) : Y \supseteq A \in \mathcal{A}\},$$

for any $Y \subseteq X$.

Note that if $\mathcal{L} \subseteq \mathcal{A}$ then there is nothing to prove. Suppose now that $\mathcal{L}$ is a lattice generated by $\mathcal{K}$ and an additional set $L_0 \notin \mathcal{A}$. Then every $L \in \mathcal{L}$ is of the form

$$L = (K_1 \cap L_0) \cup K_2 \text{ for some } K_1, K_2 \in \mathcal{K}.$$

Write $\mathcal{A}_1$ for the algebra generated by $\mathcal{A} \cup \{L_0\}$. We can extend μ to an additive measure μ_1 on $\mathcal{A}_1$ by the formula (as in L7/P7)

$$\mu_1((A \cap L_0) \cup (B \cap L_0^c)) = \mu^*(A \cap L_)) + \mu_*(B \cap L_0^c).$$

We check that μ_1 is $\mathcal{L}$-regular. Note that $X \setminus L_0$ was given the least possible measure and thanks to that

$$\mu_1(X \setminus L_0) = \sup\{\mu(K) : K \in \mathcal{K}, K \subseteq X \setminus L_0\}.$$

Then we use the usual trick: The family of those sets B for which μ_1 satisfies the regularity condition on both B and $X \setminus B$ form an algebra of sets.

The general case follows from the Kuratowski-Zorn lemma; in other words, we extend μ to an $\mathcal{L}$-regular measure on the algebra generated by $\mathcal{A} \cup \mathcal{L}$, adding new sets one by one. □

The main use of the regularity condition is that it may give countable additivity.

Lemma 19.3. *Let $\mu \in P(\mathcal{A})$ be an $\mathcal{L}$-regular for some lattice $\mathcal{L} \subseteq \mathcal{A}$. Suppose that $\lim_n \mu(L_n) = 0$ for every decreasing sequence of $L_n \in \mathcal{L}$ with empty intersection. Then μ is continuous at $\emptyset$ (and so extends to a countably additive measure on $\sigma(\mathcal{A})$).*

Proof. We need to check that $\lim_n \mu(A_n) = 0$ whenever $A_n \in \mathcal{A}$ form a decreasing sequence with empty intersection. Fix any $\varepsilon > 0$; using regurality choose $L_n \in \mathcal{L}$ such that $L_n \subseteq A_n$ and $\mu(L_n) > \mu(A_n) - \varepsilon/2^n$. Then $K_n = L_1 \cap \ldots L_n \in \mathcal{L}$ is a decreasing sequence with empty intersection so $\mu(K_n) \to 0$. But it is routine to check that $\mu(A_n \setminus K_n) < \varepsilon$ so $\mu(A_n) < 2\varepsilon$ for large n. □

20. Borel extensions of Baire measures

We can present some topological consequences of the results from the previous section. In a topological space X we have two natural lattices: the lattice $\mathcal{F}_X$ of all closed subsets of X and $\mathcal{Z}_X$ — the lattice of zero sets (see the problem list).

For some results we need separation axioms for topological spaces. Recall that a topological space X is *regular* if whenever $x \in U \subseteq X$, where U is open, there is open V such that $x \in V \subseteq \overline{V} \subseteq U$. The axiom of complete regularity requires the existence (in such a case) of a continuous function $g : X \to [0,1]$ such that $g(x) = 1$ and $g|(X \setminus U) \equiv 0$ (or vice versa). It follows that in a completely regular topological space the family of cozero sets is a base for the topology – note that the set $V = \{x \in X : g(x) > 1/2\}$ is cozero. Recall finally, that complete regularity is productive; in particular, an arbitrary product of metrizable spaces is completely regular (but does not have to be normal).

Lemma 20.1. *Every Baire measure on X is $\mathcal{Z}_X$-regular.*

Proof. We have seen the proof — it is almost the same as the argument showing that every Borel measure on a metrizable space is regular. One has to note that metrizability was used only to check that every closed set is G_δ. Here we deal with closed G_δ sets from $\mathcal{Z}_X$. □

Definition 20.2. We say that a Borel measure ν on a space X is τ-additive if $\mu(\bigcup \mathcal{U}) = \sup_{U \in \mathcal{U}} \mu(U)$ for every directed family of open sets $\mathcal{U}$.

Lemma 20.3. *Every τ-additive Borel measure on a regular topological space X is $\mathcal{F}_X$-regular.*

Every Borel measure on a compact space is τ-additive.

Proof. For the first statement take open $U \subseteq X$. By the separation axiom, for every $x \in U$ there is open V_x such that $x \in V_x \subseteq \overline{V_x} \subseteq U$. By τ-additivity of μ, for $\varepsilon > 0$ there is a finite collection $x_1, \ldots, x_n \in U$ such that for putting $W = \bigcup_{i \le n} V_{x_i}$ we have $\mu(W) > \mu(U) - \varepsilon$. Then $\overline{W} \subseteq U$ and $\mu(\overline{W}) \ge \mu(W) > \mu(U) - \varepsilon$. This shows that μ is closed-regular on U. The rest is as always: check that the family of those A for which μ is regular on A and $X \setminus A$ is a σ-algebra.

The second statement is obvious (by compactness). □

Theorem 20.4. *Let X be a completely regular space X. Every τ-additive probability Baire measure on X can be uniquely extended to a (closed-regular) τ-additive Borel measure.*

Proof. First uniqueness. Let μ be a Baire measure on X and let ν be some extension of μ to a closed-regular Borel measure. Take a closed set $F \subseteq X$. For every $x \in X \setminus F$ there is a continuous function $g : X \to [0,1]$ such that $g(x) = 0$ and $g|F \equiv 1$ (this is the axiom of completely regular topological space). Set $V_x = g^{-1}[[0,1/2)]$; then V_x is a cozero set and $V_x \cap F = \emptyset$. We have $X \setminus F = \bigcup_{x \in X \setminus F} V_x$ so by τ-additivity of ν, given $\varepsilon > 0$, there is a finite union W of such V_x's such that $\mu(W) = \nu(W) > \nu(X \setminus F) - \varepsilon$. This shows that $\nu(F) = \mu^*(F)$ so $\mu(F)$ is uniquely determined. By regularity, ν is uniquely determined on all Borel sets.

Then existence. We start from μ defined on $Baire(X)$ and use Theorem 19.2 to extend μ to a finitely additive closed-regular measure on an algebra $\mathcal{B}$ generated by $Baire(X)$ and all closed sets.

We now check that ν satisfies the condition that if the closed F_n form a decreasing sequence with empty intersection then $\lim_n \nu(F_n) = 0$.

Fix $\varepsilon > 0$; for every $x \in X$ we have $x \notin F_n$ for some n and, as above, conclude that there is a cozero set $V_x \ni x$ such that $V_x \cap F_n = \emptyset$. This defines a cover of X by sets

V_x. Since the Baire measure μ is τ-additive, there is k and $x_1, \ldots, x_k$, such that, writing $V = V_{x_1} \cup \ldots V_{x_k}$ we have $\mu(V) > 1 - \varepsilon$. Each V_{x_i} is disjoint from some F_n and therefore $V \cap F_n = \emptyset$ if n is large enough. Consequently, $\nu(F_n) < \varepsilon$ for large n.

By Lemma 18.4, ν is continuous at $\emptyset$ and extends to a Borel measure. □

Corollary 20.5. *Every product measure on a product $\prod_t X_t$ of separable metrizable spaces extends uniquely to a τ-additive Borel measure.*

Proof. Such a product measure is τ-additive Baire measure by Theorem 18.4 so we can apply Theorem 20.4. □

Corollary 20.6. *Every Baire measure on a compact topological space extends uniquely to a closed-regular Borel measure.*

Proof. If K is compact then automatically every μ on $Baire(K)$ is τ-additive. □

The last corollary explains the following puzzle. The general form of the Riesz representation theorem is that every continuous functional on the Banach space $C(K)$ with K compact is represented by integration with respect to the unique **Borel** measure. However, to integrate continuous functions we need only the measure to be defined on $Baire(K)$.