The Cantor set

The Cantor set is $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$, the space of all infinite sequences of zeros and ones.

Facts.

(1) $2^{\mathbb{N}}$ is a compact space in the product topology.

(2) $2^{\mathbb{N}}$ is metrizable, for instance $d(x,y) = 1/k$, where $k = \min\{n : x(n \neq y(n)\}$ for $x \neq y$ is a compatible metric.

(3) $2^{\mathbb{N}}$ is **zerodimensional**, i.e. it has a base of clopen sets.

(4) $h : 2^{\mathbb{N}} \to C \subseteq [0,1]$, $h(x) = \Sigma_{n=1}^{\infty} 2x(n)/3^n$ is a homeomorphism with the usual ternary Cantor set C.

(5) $f : 2^{\mathbb{N}} \to [0,1]$, $f(x) = \Sigma_{n=1}^{\infty} x(n)/2^n$ is a continuous surjection.

(6) Every compact metric space is a continuous image of $2^{\mathbb{N}}$.

Ad (6) there is a cont. surjection

$$F : 2^{\mathbb{N}} \xrightarrow[\text{onto}]{} [0,1]^{\mathbb{N}}$$

$$F: \quad 2^{\mathbb{N}} \longleftrightarrow (2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow[\text{onto}]{} [0,1]^{\mathbb{N}}$$

$(2^{\mathbb{N}})^{\mathbb{N}} \simeq 2^{\mathbb{N}\times\mathbb{N}}$ is $2^{\mathbb{N}}$

$$(x_1, x_2, \dots) \longrightarrow (f(x_1), f(x_2), \dots)$$

$x_i \in 2^{\mathbb{N}}$

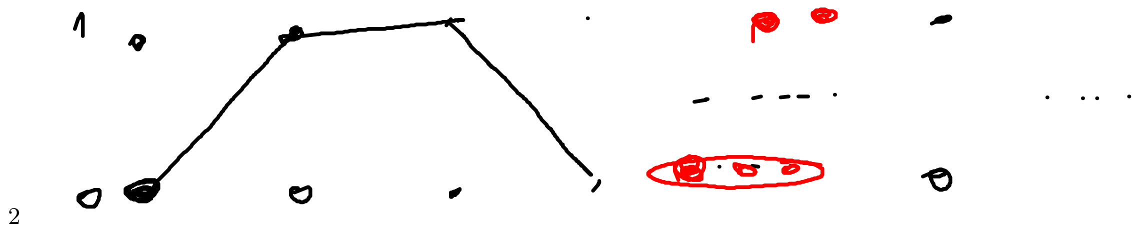
Every compact metric space $K \hookrightarrow [0,1]^{\mathbb{N}}$

$$F : 2^{\mathbb{N}} \xrightarrow[\text{onto}]{} [0,1]^{\mathbb{N}} \supseteq K$$

$A = F^{-1}[K] \subseteq 2^{\mathbb{N}}$, A closed

There is a retraction $r : 2^{\mathbb{N}} \xrightarrow[\text{onto}]{} A$ $\Rightarrow$ see P1/L2

Then $F \circ r : 2^{\mathbb{N}} \xrightarrow[\text{onto}]{} K$.



The Cantor set

Notation. *If* $I \subseteq \mathbb{N}$ *and* $\varphi : I \to \{0,1\}$ *then we write*

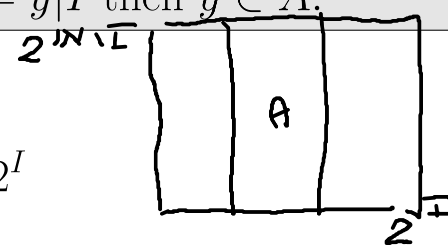
$$[\varphi] = \{x \in 2^{\mathbb{N}} : x|I = \varphi\}.$$

Remark. Sets of the form $[\varphi]$, where the domain of φ is finite, form a base of the topology on $2^{\mathbb{N}}$, and every $[\varphi]$ is clopen.

as $[\varphi]^c = \bigcup_{\psi} [\psi]$ $\quad \psi \neq \varphi \quad \psi : I \to \{0,1\}$

Definition. Given $I \subseteq \mathbb{N}$ and $A \subseteq 2^{\mathbb{N}}$, we say that A is determined by coordinates in I and write $\underline{A \sim I}$ if

$$(\forall x \in A)(\forall y \in 2^{\mathbb{N}}) \text{ if } x|I = y|I \text{ then } y \in A.$$



Equivalently, $A \sim I$ means that

$$A = \pi_I^{-1}[\pi_I[A]], \quad \pi_I : 2^{\mathbb{N}} \to 2^I$$

is the projection.

Lemma. $C \subseteq 2^{\mathbb{N}}$ *is clopen if and only if* C *depends on finitely many coordinates.* $(\exists I \subseteq \mathbb{N}$ finite $\quad C \sim I)$.

Proof.

Assume C is clopen. Then is a union of finitely many basic sets.

$$C = \bigcup_{j=1}^{k} [\varphi_j] \qquad \varphi_j : I_j \to \{0,1\}$$

I_j finite.

$I = \bigcup_j I_j$ is finite. Then $C \sim I$.

$2^{\mathbb{N}}$ as a topological group

$\{0,1\}$ is a group with the operation $a \oplus b = a + b \mod 2$. So is $2^{\mathbb{N}}$ with the coordinatewise addition mod 2:

$$x \oplus y = (x(1) \oplus y(1), x(2) \oplus y(2), \ldots).$$

$-x = x$

Definition. A topological group G is a group equipped with some topology for which

$$G \times G \ni (x, y) \mapsto x \cdot y \in G, \quad G \ni x \mapsto x^{-1} \in G$$

are continuous.

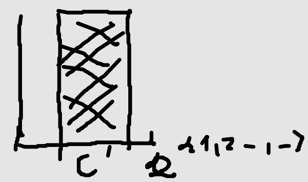
Fact. $(2^{\mathbb{N}}, \oplus)$ *is a compact topological group.*

$$\nu = \bigotimes_{n=1}^{\infty} \frac{1}{2}(\delta_0 + \delta_1)$$

The measure on $2^{\mathbb{N}}$

Definition. If $C \sim \{1, 2, \ldots, n\}$ then we can write $C = C' \times \{0,1\} \times \{0,1\} \ldots$. Put

$$\nu(C) = \frac{|C'|}{2^n}.$$



Theorem. ν is properly defined finitely additive set-function on the algebra $\mathrm{Clop}(2^{\mathbb{N}})$ of clopen sets.
Such ν is continuous from above and therefore has a <u>unique</u> extension to a measure on $Bor(2^{\mathbb{N}}) = \sigma(\mathrm{Clop}(2^{\mathbb{N}}))$.
(*) For every $B \in Bor(2^{\mathbb{N}})$ and $\varepsilon > 0$ there is $C \in \mathrm{Clop}(2^{\mathbb{N}})$ such that $\nu(B \triangle C) < \varepsilon$.

Continuous from above at $\emptyset$ $\quad A_n \in \mathrm{Clop}(2^{\mathbb{N}})$

$A_1 \supseteq A_2 \supseteq \ldots \quad \bigcap A_n = \emptyset \Longrightarrow \lim \nu(A_n) = 0$

(*) $\mathcal{A} = \{B \in Bor(2^{\mathbb{N}}) : \forall \varepsilon\ \exists C \in \mathrm{Clop}(2^{\mathbb{N}})\ \nu(B \triangle C) < \varepsilon\}$

$\mathrm{Clop}(2^{\mathbb{N}}) \subseteq \mathcal{A}$ and $\mathcal{A}$ is a σ algebra

$\mathcal{A} = Bor(2^{\mathbb{N}})$

The Haar measure

Theorem. The measure ν is the Haar measure of the compact group $2^{\mathbb{N}}$, i.e. ν is the unique probability measure which is translation invariant, $\nu(x \oplus B) = \nu(B)$ for every $x \in 2^{\mathbb{N}}$ and $B \in Bor(2^{\mathbb{N}})$.

$x \oplus B = \{x \oplus b,\ b \in B\}$

Sketchy Proof:

- $\varphi : I \to \{0,1\}$ $\quad [\varphi] = \{x \in 2^{\mathbb{N}} : x|I = \varphi\}$

  $\nu([\varphi]) = \frac{1}{2^{|I|}}$

  $x \oplus [\varphi] = [\psi] \quad \psi(n) = x(n) \oplus \varphi(n) \quad n \in I.$

  $\nu(x \oplus [\varphi]) = \nu([\varphi]).$

- $C \in Clop(2^{\mathbb{N}}) \longrightarrow \nu(x \oplus C) = \nu(C).$

- We want $\nu(x \oplus B) = \nu(B)$ for $B \in Bor(2^{\mathbb{N}})$

  Define ν' on $Bor(2^{\mathbb{N}})$

  $\nu'(B) = \nu(x \oplus B)$

  Then $\nu' | Clop(2^{\mathbb{N}}) = \nu | Clop(2^{\mathbb{N}})$

  so $\nu' = \nu$.

Basic probability in $2^{\mathbb{N}}$

Definition. A set $A \subseteq 2^{\mathbb{N}}$ is called a tail set (*zbiór resztowy*) if $A \sim \{k : k \geqslant n\}$ for every n. In other words, A does not depend on finite number of coordinates, that is if $a \in A$ and $x(n) = a(n)$ for almost all n then $x \in A$.

Example. There are natural example of tail sets, for instance

$$A(\beta) = \left\{ x \in 2^{\mathbb{N}} : \lim_{n\to\infty} \frac{x(1) + x(2) + \ldots x(n)}{n} = \beta \right\},$$

is a tail set since changing finite number of coordinates does affect the limit.

Kolmogorov's 0-1 law

Theorem. *If $A \in Bor(2^{\mathbb{N}})$ is a Borel tail set then $\nu(A) = 0$ or $\nu(A) = 1$.*

Proof. Consider a finite $I \subseteq \mathbb{N}$ and $\varphi : I \to 2$. Take n such that $I \subseteq \{1, 2, \dots, n\}$; then $[\varphi]$ depends on first n coordinates while A depends on $\{n+1, n+2, \dots\}$. Hence $\nu([\varphi] \cap A) = \nu([\varphi]) \cdot \nu(A)$, see L2/P3[1].

Every $C \in clop(2^{\mathbb{N}})$ is a finite disjoint union of such sets $[\varphi]$; it follows easily that $\nu(C \cap A) = \nu(C) \cdot \nu(A)$ for any clopen set C.

Suppose that $\nu(A) > 0$; we shall check that $\nu(A) = 1$. Take $\varepsilon > 0$ and choose $C \in clop(2^{\mathbb{N}})$ such that $\nu(A \triangle C) < \varepsilon \cdot \nu(A)$. Then

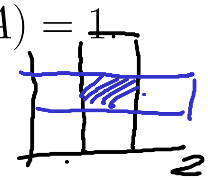
$$\nu(A) \cdot \nu(C) = \nu(A \cap C) \geqslant \nu(A) - \varepsilon \cdot \nu(A),$$

which gives $\nu(C) \geqslant 1 - \varepsilon$. Hence

$$\nu(A) \geqslant \nu(C) - \varepsilon \geqslant 1 - 2\varepsilon;$$

as ε may be arbitrarily small, we get $\nu(A) = 1$.

2^J



2^I

L2/P3: $A, B \in Bor(2^{\mathbb{N}})$

$A \sim I$

$B \sim J$

$I \cap J$

$\longrightarrow \nu(A \cap B) = \nu(A) \cdot \nu(B)$

[1] List 2/Problem 3

Normal numbers

$$A(\alpha) = \left\{x \in 2^{\mathbb{N}} : \lim_{n\to\infty} \frac{x(1) + x(2) + \dots x(n)}{n} = \alpha\right\}$$

We have $\nu(A(\alpha)) \in \{0, 1\}$. Note that $\nu(A(\alpha)) = \nu(A(1-\alpha))$ which may suggest that $\nu(A(1/2)) = 1\dots$

Theorem of Borel on normal numbers.

$$\boxed{\nu(A(1/2)) = 1.}$$

Follows from SL of LN :

Proof. Fix $\alpha < 1/2$ and set

$$B_n^\alpha = \{x \in 2^{\mathbb{N}} : \frac{x(1) + \dots x(n)}{n} \leqslant \alpha\}.$$

Claim. There is $\theta < 1$ such that $\nu(B_n^\alpha) \leqslant \theta^n$ for every n.

$= \{x \in 2^{\mathbb{N}} : \lim_{n\to\infty} \frac{x(1)+..+x(n)}{n} \leq \alpha\}$

By Claim, if $\boxed{B^\alpha = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty B_k^\alpha,}$ then $\nu(B^\alpha) = 0$ for every $\alpha < 1/2$. This means that the set of those $x \in 2^{\mathbb{N}}$ for which

$$\liminf_{n\to\infty} \frac{x(1) + \dots x(n)}{n} < 1/2 - \delta,$$

has measure zero for every $\delta > 0$. Finally, the same holds for $\delta = 0$.

It follows, by a symmetric argument, that the set of those $x \in 2^{\mathbb{N}}$ for which

$$\limsup_{n\to\infty} \frac{x(1) + \ldots x(n)}{n} > 1/2,$$

has also measure zero, and we are done.

Using L2/P 10, we can conclude from that for λ-almost all $x \in [0,1]$, x has a uniform distribution of '0' and '1' in its binary expansion. This is the simplest form of Borel's theorem, see e.g.
`https://en.wikipedia.org/wiki/Normal_number`
`https://www.emis.de/journals/AUSM/C2-1/math21-8.pdf`
for further discussion.

Claim

$$B_n^\alpha = \{x \in 2^{\mathbb{N}} : \frac{x(1) + \ldots x(n)}{n} \leqslant \alpha\}, \quad \alpha < 1/2.$$

Claim. There is $\theta < 1$ such that $\nu(B_n^\alpha) \leqslant \theta^n$ for every n.

Note that $\nu(B_n^\alpha) = c_n/2^n$, where

$$c_n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{[n\alpha]};$$

We are to prove that there is $\theta < 1$ such that $c_n/2^n \leqslant \theta^n$ for every n.

Note that for any $t \in (0,1)$ we have

$$t^{\alpha n} \cdot c_n \leqslant \binom{n}{0} + \binom{n}{1} t + \ldots + \binom{n}{[\alpha n]} t^{[\alpha n]} \leqslant (1+t)^n,$$

so

$$c_n \leqslant \left(\frac{1+t}{t^\alpha}\right)^n.$$

It remains to find t such that $(1+t)/t^\alpha < 2$. For this consider $f(t) = 2t^\alpha - t - 1$: we have $f(1) = 0$ and $f'(1) = 2\alpha - 1 < 0$ so there is $t < 1$ such that $f(t) > 0$, as required.