3. Intro on metrizable spaces

We shall discuss some properties of separable metrizable spaces. Such spaces have a countable base so, by a general topological fact, can be embedded into $[0,1]^{\mathbb{N}}$. In fact this can be proved directly:

Theorem 3.1. *If X is separable metrizable space then X is homeomorphic to a subspace of $[0,1]^{\mathbb{N}}$.*

Proof. (Sketch) Take a metric ρ on X bounded by 1. Let $\{x_n : n \in \mathbb{N}\}$ be a dense set in X. For every n, $f_n(x) = \rho(x, x_n)$ defines a continuous function $f_n : X \to [0,1]$. It suffices to check that $F : X \to [0,1]^{\mathbb{N}}$ where $F(x) = (f_n(x))_n$, is a homeomorphic embedding. $\square$

There are many linear spaces that are metrizable. Usually, we first define on a linear space X a norm $\|\cdot\|$, i.e. a function $X \to \mathbb{R}_+$ such that $\|x\| = 0$ iff $x = 0$, $\|a \cdot x\| = |a|\|x\|$ and $\|x+y\| \le \|x\|+\|y\|$ for any $x, y \in X$ and $a \in \mathbb{R}$. Then $\rho(x,y) = \|x-y\|$ defines a metric on X. This is so in case of Euclidean spaces $\mathbb{R}^d$: the Euclidean norm is $\|x\| = \sqrt{\sum_{i=1}^d x_i^2}$.

The same is done for many 'infinitely dimensional' spaces; for instance $L_1[0,1]$ is given the norm $\|f\|_1 = \int_0^1 |f| \; \mathrm{d}\lambda$. Here $\|\cdot\|_1$ is indeed a norm; if $\|f\|_1 = 0$ then $f = 0$ almost everywhere, i.e. $f = 0$ in $L_1[0,1]$, where we identify functions equal almost everywhere.

Example 3.2. For any topological space we write $C_b(X)$ for the space of real-valued bounded continuous functions on X. This space is given the norm $\|\cdot\|_\infty$, where $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Note that convergence in the sup-norm is simply uniform convergence. The space $C_b(X)$ is complete in this norm (recall that *uniformly convergent sequence of continuous functions has a continuous limit*).

Note that in case of a compact space K, $C_b(K)$ is denoted simply by $C(K)$, since every continuous functions on K is bounded Recall the following important fact.

Theorem 3.3 (Stone-Weierstrass)**.** *Let K be a compact space. Suppose that $W \subseteq C(K)$ is a linear subspace which is also a ring, i.e. W is closed under multiplication.*

If W contains constant functions and distinguishes points of K (i.e. for different $x, y \in K$ there is $w \in W$ such that $w(x) \neq w(y)$) then W is dense in $C(K)$, so every continuous function on K can be uniformly approximated by elements of W..

Proof. Let us believe in it! Engelking's book gives an elegant argument. Those interested may *google* other proofs. $\square$

Theorem 3.4. *Let X be a metric space. Then $C_b(X)$ is separable if and only if X is compact.*

Proof. Consider first $K = [0, 1^{\mathbb{N}}$. Let $W \subseteq C(K)$ be the subspace of all linear combinations of functions that are either constant or are of the form

$$f_{n_1,n_2,\ldots,n_k} = \pi_{n_1} \cdot \pi_{n2} \cdot \ldots \cdot \pi_{n_k},$$

where $k \in \mathbb{N}$ and π_n denotes the projection onto the nth coordinate. Then W is dense in $C(K)$ by 3.3 — checking that W satisfies the assumptions should be easy. It is almost immediate, that rational combinations of functions like $f_{n_1,n_2,\ldots,n_k}$ form a countable set W_0 which is dense in W so also in $C(K)$.

Now if X is compact then we may assume that X is a closed subset of K above. Note that $C(X) = \{f|X : f \in C(K)$, because by Tietze's extension theorem every continuous funcion on X extends to a continuous function on K. It follows from the above, that taking the restriction to K of functions from W_0 we get a countable dense set in $C(X)$.

Suppose now that X is not compact. Since X is metrizable, there is a sequence of $x_n \in X$ without converging subsequences. Then the set $F = \{x_n : n \in \mathbb{N}\}$ is closed in X. It is also discrete (every point of F is isolated) so every function $g : F \to 2$ is continuous. For two different such functions f, g we have $\|f - g\|_\infty = 1$. It follows that $C_b(F)$ is not separable. (*why?*) Again $C_b(F)$ agrees with $\{f|F : f \in C_b(X)\}$ (by Tietze's theorem) so $C_B(X)$ is not separable either. $\square$

Definition 3.5. A topological space is *Polish* if it is separable and metrizable by a complete metric.

Euclidean space $\mathbb{R}^d$ are clearly Polish, so are $\mathbb{R}^{\mathbb{N}}$ and $[0, 1]^{\mathbb{N}}$. If X is a Polish space and Y is a closed subspace of X then Y is Polish too: if ρ is a complete metric on X then its restriction to Y is also complete (recall also that a subspace of a metrizable separable space is itself separable). Recall that a G_δ set is one that can be written as a countable intersection of open sets.

Theorem 3.6 (Alexandrov's)**.** *A subspace Y of a Polish space X is itself Polish if and only if Y is a G_δ subset of X.*

Proof. See L2/P B. $\square$

Note that 3.6 implies that $\mathbb{R} \setminus \mathbb{Q}$ is Polish! Do not be shocked, the usual Euclidean metric is *veery* incomplete on irrationals but there is an equivalent compete metric on $\mathbb{R} \setminus \mathbb{Q}$. The space $\mathbb{R} \setminus \mathbb{Q}$ has a special role in descriptive set theory; this will be mentioned later.

4. Borel sets

For the time being, we consider always separable metrizable spaces $X, Y, \ldots$**.**

Given such a space X, we define the Borel σ-algebra $Bor(X)$ to be the smallest σ-algebra containing all open subsets of X; if $\mathcal{T}_X$ denotes the topology on X, we can write $Bor(X) = \sigma(\mathcal{T}_X)$.

There is a fine theory of Borel structures. see [Kechris, 11.B] but we need only few basic facts. Traditionally, a set $A \subseteq X$ is called a G_δ set if $A = \bigcap_n V_n$ for some open V_n, and is called an F_σ set if $A = \bigcup_n F_n$ for some closed sets F_n. Note that A is G_δ in X if and only if $X \setminus A$ is F_σ.

Lemma 4.1. *Every closed $F \subseteq X$ is G_δ (so every open set if F_σ).*

Proof. Let ρ be a compatible metric on X. Given a closed set $F \subseteq X$, define

$$V_n = \{x \in X : \rho(x, F) < 1/n\}.$$

Then V_n is open since the function $x \to \rho(x, F)$ is continuous. If $x \in \bigcap_n V_n$ then $\rho(x, F) = 0$ so $x \in F$ (as F is closed). □

Lemma 4.2. *If $Y \subseteq X$ then $Bor(Y) = \{B \cap Y : B \in Bor(X)\}$.*

Proof. Exercise! □

Theorem 4.3. *For any separable metrizable spaces X, Y we have*

$$Bor(X \times Y) = Bor(X) \otimes Bor(Y) := \sigma(\{A \times B : A \in Bor(X), B \in Bor(Y)\}).$$

Proof. Let us check the only nontrivial thing, that if $U \subseteq X \times Y$ is open then $U \in Bor(X) \otimes Bor(Y)$. Take countable bases $\mathcal{V}_X$ and $\mathcal{V}_Y$ in X and Y. Then

$$U = \bigcup \{V \times V' : V \in \mathcal{V}_X, V' \in \mathcal{V}_Y, V \times V' \subseteq V\},$$

so U is a countable union of open rectangles and thus $U \in Bor(X) \otimes Bor(Y)$. □

Theorem 4.4. *For any separable metrizable spaces X_n we have*

$$Bor(\prod_n X_n) = \bigotimes_n Bor(X_n).$$

Proof. By definition, the σ-algebra on RHS is the one generated by all the Borel rectangles

$$B_1 \times B_2 \times \ldots \times B_n \times X_{n+1} \times \ldots.$$

Again, the only fact that we need to check is that if $U \subseteq \prod_n X_n$ is open then U can be written as a countable union of open rectangle.

Looking back at the previous proof, it should be clear that we can repeat the argument, as the product space has a base consisting of 'open finite-dimensional rectangles'. □

Remark 4.5. Theorem 4.3 does not hold for merizable spaces that are not separable: take a discrete X of size $> \mathfrak{c}$ and check that the diagonal in $X \times X$ is not in $Bor(X) \otimes Bor(X)$. We shall come back to this later.

5. Borel measures

Given the space X, we write $P(X)$ for the family of all probability measures defined on $Bor(X)$, called simply *Borel measures on* X. [1]

Theorem 5.1. *For any X, every $\mu \in P(X)$ is* **regular***, i.e. for every $B \in Bor(X)$ and every $\varepsilon > 0$ there are an open set V and a closed set F such that $F \subseteq B \subseteq V$ and $\mu(V \setminus F) < \varepsilon$.*

Proof. Consider the family $\mathcal{A}$ of those $A \in Bor(X)$ which have the required property (can be approximated from below and from above as stated). Note first that every open $U \subseteq X$ belongs to $\mathcal{A}$: indeed, using Lemma 4.1 we can write $U = \bigcup_n F_n$ for some closed sets F_n. Taking $F_1, F_1 \cup F_2, \ldots$, we can assume that, actually, F_n are increasing. By the continuity of the measure, $\mu(F_n) \to \mu(U)$. Of course, U is approximated from above by itself.

Now it remains to check that $\mathcal{A}$ is a σ-algebra because we then have $\mathcal{A} = Bor(X)$, and this is what we want. Note that if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$, by symmetry of the condition. Consider $A_n \in \mathcal{A}$ and set $A = \bigcup_n A_n$. Take $F_m \subseteq A_n \subseteq V_n$, where F_n are closed, V_n are open and $\mu(V_n \setminus F_n) < \varepsilon/2^{n+1}$. Put $V = \bigcup_n V_n$ and $C = \bigcup_n F_n$; consider also $C_N = \bigcup_{n \le N} F_n$. Then $\mu(C_N) \to \mu(C)$ so $\mu(C \setminus C_N) < \varepsilon/2$ if N is large enough. Then

$$\mu(V \setminus C_N) \le \mu(V \setminus C) + \mu(C \setminus C_N) \le \sum_n \varepsilon/2^{n+1} + \varepsilon/2 = \varepsilon.$$

Now $C_N \subseteq A \subseteq V$, C_N is closed, V is open, as required. $\square$

Remark 5.2. We can distinguish inner-regularity and outer-regularity defined in an obvious manner. Note that inner-regularity implies the other (and vice versa); however, it would be more difficult to check, say, inner-regularity alone!

Corollary 5.3. *If μ is a Borel measure on X then for every $B \in Bor(X)$ there are an F_σ set A_1 nd a G_δ set A_2 such that $A_1 \subseteq B \subseteq A_2$ and $\mu(A_2 \setminus A_1) = 0$.*

Proof. Apply 5.1 for $\varepsilon = 1/n$ to get F_n's and V_n's; then put $A_1 = \bigcup_n F_n$ and $A_2 = \bigcap_n V_n$. $\square$

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Working with measures, we do not care much about measure zero sets. The above fact explains that from that point of view we loose interest in the Borel hierarchy at the $G_\delta - F_\sigma$ lever.

Theorem 5.4. *Let X be a Polish space. Then every $\mu \in P(K)$ is* **tight***, i.e. for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(K) \ge 1 - \varepsilon$.*

Proof. Take a compatible complete metric ρ on X; we write $B(x, r)$ for the open ball $\{y \in X : \rho(x, y) < r\}$. Let $D \subseteq X$ be a countable dense set

[1] P for **p**robability; we need sometimes to refer to the power set of X, then we write $\mathcal{P}(X)$.

Fix $\varepsilon > 0$; for every n we have $X = \bigcup_{d\in D} B(d, 1/n)$ (as D is dense). Hence there is a finite $I_n \subseteq D$ such that

$$\mu(\bigcup_{d\in I_n} B(d, 1/n)) > 1 - \varepsilon/2^n.$$

Put

$$F_n = \overline{\bigcup_{d\in I_n} B(d, 1/n)},$$

and let $K = \bigcap_n F_n$. We have $\mu(X \setminus F_n) \le \varepsilon/2^n$, so

$$\mu(X \setminus K) \le \sum_n \varepsilon/2^n = \varepsilon.$$

It remains to check that K is compact. This follows from the fact that K is a closed set in a complete metric space (X, ρ), and K is totally bounded (for every $\delta > 0$, K has a finite δ-net). This is clear in view of the very definition of K. □

Corollary 5.5. *If X is Polish then for every $\mu \in P(X)$, any $B \in Bor(X)$ and $\varepsilon > 0$ there is a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$.*

Proof. By 5.1, there is closed $F \subseteq B$ with $\mu(B \setminus F) < \varepsilon/2$. By 5.4 there is compact $K_1 \subseteq X$ with $\mu(X \setminus K_1) < \varepsilon/2$. Take $K = F \cap K_1$. □

A function $f : X \to \mathbb{R}$ is *Borel* if $f^{-1}[B] \in Bor(X)$ for every $B \in Bor(\mathbb{R})$. The following is sometimes called Lusin's theorem.

Theorem 5.6 (Lusin)**.** *If $f : X \to \mathbb{R}$ is a Borel function and $\mu \in P(K)$ then for every $\varepsilon > 0$ there is a closed set $F \subseteq X$ such that $\mu(X \setminus F) < \varepsilon$ and the restriction $g|F : F \to \mathbb{R}$ is continuous.*

If X is Polish, we can find such a set F which is moreover compact.

Proof. Note first that if $g = \chi_B$ is the characteristic function of some $B \in Bor(X)$ then this is simple: take closed sets $F_1 \subseteq B$ and $F_2 \subseteq X \setminus B$ such that $\mu(X \setminus (F_1 \cup F_2)) < \varepsilon$ (use 5.1).Then $F = F_1 \cup F_2$ is as required (we mean: g is continuous on the subspace F of X).

Consider a Borel simple function $g = \sum_{k\le n} a_k \chi_{B_k}$, where B_k are pairwise disjoint. We repeat the above trick: take closed $F_k \subseteq B_k$ and closed $F_0 \subseteq X \setminus \bigcup_{k\le n} F_n$ so that $F = F_0 \cup \ldots \cup F_n$ satisfies $\mu(X \setminus F) < \varepsilon$. Again, g is continuous on F,

Consider now any Borel function g. Then there is $B_0 \in Bor(X)$ such that $\mu(X \setminus B_0) < \varepsilon$ and g is bounded on B_0. Recall that $g|B$ is then a uniform limit of a sequence of simple Borel functions $g_n : B_0 \to \mathbb{R}$. For every n find a closed set $F_n \subseteq B_0$ so that $g_n|F_n$ is continuous and $\mu(B_0 \setminus F_n) < \varepsilon/2^n$. Finally, set $F = \bigcap_n F_n$. Then obvious calculation give $\mu(X \setminus F) < 2\varepsilon$. Now $g_n|F$ is a sequence of continuous functions on F converging uniformly to $g|F$, so $g|F$ is continuous too.

For the proof of the final statement, repeat the trick from 5.5 □

An abstract measure space $(X, \mathcal{B}, \mu)$ is (sometimes) called *separable* if $L_1(\mu)$ is separable, it has a countable set which is dense in $\|\cdot\|_1$-norm.

Lemma 5.7. *Suppose that for a measure space $(X, \mathcal{B}, \mu)$ there is a countable family $\mathcal{A} \subseteq \mathcal{B}$ such that*

$$\inf\{\mu(A \triangle B) : A \in \mathcal{A}\} = 0,$$

for every $B \in \mathcal{B}$. Then the measure μ is separable.

Proof. The recipe for a countable dense set in $L_1(\mu)$: take all simple functions

$$\sum_{k \le n} a_k \chi_{A_k}, \text{ where } a_k \in \mathbb{Q}, A_k \in \mathcal{A}, n \in \mathbb{N}.$$

It is routine to check that such a set is dense in the subspace of all simple functions and so is dense in $L_1(\mu)$. □

Theorem 5.8. *If X is separable metrizable then every $\mu \in P(K)$ is separable (in the above sense).*

Proof. Take a countable base $\mathcal{V}$ of X. Then the family

$$\mathcal{U} = \{V_1 \cup \ldots \cup V_n : n \in \mathbb{N}, V_i \in \mathcal{V}\},$$

is also countable and every open set $W \subseteq X$ is an increasing union of elements of $\mathcal{U}$.

If $B \in Bor(X)$ and $\varepsilon > 0$ then, by 5.1, there is open $W \supseteq B$ with $\mu(W \setminus B) < \varepsilon/2$. Then (by above) there is $U \in \mathcal{U}$ such that $U \subseteq W$ and $\mu(W \setminus U) < \varepsilon/2$. Then

$$\mu(U \triangle B) = \mu(U \setminus B) + \mu(B \setminus U) \le \mu(W \setminus B) + \mu(W \setminus U)\ \varepsilon/2 + \varepsilon/2 = \varepsilon;$$

we can apply Lemma 5.7. □