1. Let $\mu \in P(X)$. Recall that we define $\mu^*(Z) = \inf\{\mu(B) : B \in Bor(X), B \supseteq Z\}$ for any $Z \subseteq X$.

Check that the formula $\nu(B \cap Y) = \mu^*(B \cap Y)$ defines a measure $\nu \in P(Y)$ for any $Y \subseteq X$.

2. Let $\mu \in P(X)$. Check that for an arbitrary set $Z \subseteq X$ there is $H \in Bor(X)$ (in fact, of type G_δ), such that $Z \subseteq H$ and $\mu(H) = \mu^*(Z)$. Such H, determined up to measure zero set, is called the measurable hull of Z.

3. Let X be a separable metrizable space. Check that if $\mu, \nu \in P(X)$ agree on some base closed under finite unions then $\mu = \nu$; in particular, $|P(X)| = \mathfrak{c}$ for $|X| > 1$.

4. Generalize Lusin's theorem: If $f : X \to Y$ is a Borel map between separable metrizable spaces then for every $\mu \in P(X)$ and $\varepsilon > 0$ there is a closed set $F \subseteq X$ such that $\mu(X \setminus F) < \varepsilon$ and $f|F : F \to Y$ is continuous.

HINT. Think that $Y \subseteq [0,1]^{\mathbb{N}}$.

5. Find a direct argument for the following: every $\mu \in P(\mathbb{N}^\omega)$ is tight. [1]

6. **Analytic sets.** A set A in a Polish space is said to be *analytic* if there is a continuous map $f : \mathbb{N}^\omega \to X$ such that $f[\mathbb{N}^\omega] = A$. Let

$$\mathbb{N}^{<\omega} := \bigcup_k \mathbb{N}^k,$$

denote the set of all finite sequences of natural numbers (including the empty sequence $\emptyset \in \mathbb{N}^0$). We write $\alpha|n = (\alpha_0, \ldots, \alpha_{n-1})$; $\sigma_1 \frown \sigma_2$ denotes the concatenation (putting sequences σ_1, σ_2 together.

By a *(regular) Souslin scheme* in a Polish space X we mean a family of closed sets $F = \{F_\sigma : \sigma \in \mathbb{N}^{<\omega}\}$ such that $F_{\sigma \frown n} \subseteq F_\sigma$ for every $\sigma \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$. Using such a scheme we define

$$A(F) = \bigcup_{\alpha \in \mathbb{N}^\omega} \bigcap_k F_{\alpha|k},$$

as the result of the Souslin operation.

Prove that $A \subseteq X$ is analytic if and only if A is the result of the Souslin operation over some Souslin scheme or read the proof of Thm 25.7 in [Kechris].

7. Let X be a Polish space and let $A \subseteq X$ be analytic. Prove that A is measurable with respect to every $\mu \in P(X)$ (we say that A is universally measurable).

HINT. Consider a Souslin scheme $F = \{F_\sigma : \sigma \in \mathbb{N}^{<\omega}\}$ giving A. Let H_σ be a measurable hull of the set $\bigcup_{\alpha \in \mathbb{N}^\omega} \bigcap_k F_{\sigma \frown (\alpha|k)}$, such that $H_\sigma \subseteq F_\sigma$; in particular, $H_\emptyset$ is a measurable hull of A. Check that

$$H_\emptyset \setminus A \subseteq \bigcup_{k=0}^{\infty} \bigcup_{\sigma \in \mathbb{N}^k} (H_\sigma \setminus \bigcup_{m=0}^{\infty} H_{\sigma \frown m}).$$

[1] Here the index set $\omega = \{0, 1, 2, \ldots\}$ is also the set of natural numbers:-); soon it will be convenient to count from 0

Remark. Descriptive set theory proves that every Borel set in a Polish space is analytic but there are always analytic non-Borel sets in uncountable Polish spaces.

8. Baby Selection Theorem. Let $g : K \to L$ be a continuous surjection between compact metrizable spaces. Prove that there is a Borel function $s : L \to K$ such that $f \circ s = id_L$.

Hint. Assume first that $K \subseteq [0,1]$ and define $s(y) = \inf\{x \in K : f(x) = y\}$. For the general case use the fact that K is a continuous image of the Cantor set contained in $[0,1]$.

9. If $f : X \to Y$ is a Borel map between separable metrizable spaces then for every $\mu \in P(X)$, the image measure $f[\mu] \in P(Y)$ is defined by $f[\mu](B) = \mu(f^{-1}[B])$ for $B \in Bor(Y)$. Hence, f induces the map $P(X) \to P(Y)$.

Prove that if $f : K \to L$ is a continuous surjection between metrizable compacta then $P(K) \ni \mu \to f[\mu] \in P(L)$ is also surjective.

Hint. Use Baby Selection Theorem. Actually, the fact holds for a Borel map between Polish spaces but one needs more advanced selection theorems (such as von Neumann's).