1. Check that the family of sets $A \subseteq \prod_{t\in T} X_t$ depending on countably many coordinates is a σ-algebra.

2. Check that if a subset A of the product $\prod_{t\in T} X_t$ of topological spaces depends on coordinates in $I \subseteq T$ then so does $\overline{A}$.

3. Let X be a topological space; $\mathcal{Z}_X$ denotes the family of zero sets (those of the form $g^{-1}(0)$ for a continuous function $g : X \to \mathbb{R}$).

Check that $\mathcal{Z}_X$ is a lattice of sets and, moreover, it is closed under taking countable intersections.

Dually, the family of cozero sets is a lattice closed under finite intersection and countable unions.

4. The product space $2^{\mathfrak{c}} = \{0,1\}^{\mathfrak{c}}$ is separable. To check this, replace $\mathfrak{c}$ by $[0,1]$, and consider elements of $2^{[0,1]}$ of the form χ_J, where $J \subseteq [0,1]$ is a finite union of intervals with rational endpoints.

Remark. A similar argument can be applied to show that every $\mathfrak{c}$-fold product of separable spaces is separable (Marczewski's theorem).

5. A family of sets $\mathcal{F}$ is said to be a Δ-system if there is (a root) R, such that $F \cap F' = R$ for any pair of distinct $F, F' \in \mathcal{F}$.

Prove the Δ-system lemma: *If $\mathcal{F}$ is an uncountable family of finite sets then $\mathcal{F}$ contains an uncountable $\mathcal{F}_0$ that is a Δ system.*

Hint: Assume that $|F| = n$ for every $F \in \mathcal{F}$ and apply induction on n.

6. Prove that an arbitrary product $\prod_{t\in T} X_t$ of separable metrizable spaces X_t is *ccc*.

Hint: Consider uncountably many basic open sets V_γ depending on finite sets $I_\gamma \subseteq T$; apply the Δ-system lemma to I_γ's.

7. Let ν be the usual product measure on 2^T ($1/2 - 1/2$ on each axis), where T is uncountable. Let X be the set of those $x \in 2^T$ for which $\{t \in T : x(t) = 1\}$ is countable.

Prove that X is not measurable, more precisely, $\nu_*(X) = 0$ i $\nu^*(X) = 1$.

8. Denote by ν_κ the usual product measure on 2^κ and let $\mathfrak{A}_\kappa$ be the measure algebra of ν_κ.

(a) Note that, by Kakutani's theorem, $\mathfrak{A}_\kappa$ can be seen as $Bor(2^\kappa)$ mod zero sets.

(b) Check that $|\mathfrak{A}_\kappa| = \mathfrak{c}$ for $\omega \leqslant \kappa \leqslant \mathfrak{c}$.

(c) Prove that the Boolean algebras $\mathfrak{A}_\kappa$ are pairwise nonisomorphic for different cardinal numbers κ.

Hint: The density of $\mathfrak{A}_\kappa$ (in the metric $\rho(a,b) = \nu_\kappa(a \triangle b)$) equals κ.

Remark. So, finally, there are more measures but they are obtained by forming products of the only one that is separable. *The Maharam structure theorem for measure algebras* states that, essentially, there are no more examples.

9. Let $S = (0,1)$ be the Sorgenfrey line, that is the topology on $(0,1)$ is generated by the base of sets of the form $[a,b)$, $0 < a < b \leqslant 1$. Check that $Bor(S) = Bor((0,1))$.

Hence the Lebesgue measure is a Borel measure on S. Check that λ is not tight since every compact subset of S is countable.

10. Prove that $Bor(S) \otimes Bor(S) \neq Bor(S \times S)$ for the space S from the previous problem.

Hint: $S \times S$ contains a discrete subspace of size $\mathfrak{c}$.

11. Let K and L be compact spaces and let μ and ν be closed regular Borel measures on them. Prove that $\mu \otimes \nu$ (defined on $Bor(K) \otimes Bor(L)$) extends to a Borel measure on $K \times L$.

Hint: Use the extension theorem from the lecture. Note that $\mu \otimes \nu$ is regular with respect to the lattice generated by $A \times B$ with A, B closed.

Remark. A bigger challenge is to check that if λ is the Borel extension of $\mu \otimes \nu$ then λ still satisfies the Fubini formula

$$\lambda(B) = \int_K \nu(B_x)\, \mathrm{d}\mu(x),$$

for all Borel sets $B \subseteq K \times L$.

12. Let $X = \prod_{t \in T} X_t$ be an arbitrary product of Polish spaces. Suppose that for every finite $I \subseteq T$ we are given a probability Borel measure μ_I on $X_I = \prod_{t \in I} X_t$. Suppose that the family of those μ_I is consistent in the sense that μ_I is the image of μ_J under the projection $X_J \to X_I$ whenever $I \subseteq J$.

The Kolmogorov theorem (on consistent distributions) states that there is a unique μ on $Baire(X)$ such that $\pi_I[\mu] = \mu_I$ for every finite I. Can you figure out a proof?

Hint: One can define μ on the algebra $\mathcal{A}$ of sets depending on a finite number of coordinates in an obvious way. Check that μ is $\mathcal{L}$-regular on $\mathcal{A}$, where $\mathcal{L}$ is a lattice generated by ‘finite dimensional rectangles with compact sides’; then use Lemma 19.3.