1. You work in a library and you need to buy 40 book shelves to accommodate 8450 new books. There are smaller shelves for 200 volumes at the price 260PLN per item and bigger ones for 300 volumes costing 400PLN. What will you do?

2. Observe that the above question leads to the following problem:

$$\begin{aligned} \text{minimize} \quad & 260x_1 + 400x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant 40 \\ & 200x_1 + 300x_2 \geqslant 8450 \\ & x_1, x_2 \geqslant 0, \end{aligned}$$

however, we expect a integer-valued solution.

3. You now work in a metal plant and your task is to make 1000kg of an alloy of tin, zinc and lead consisting in 40% of tin, in 35% of zinc and in 25% of lead. There are alloys of three kinds available and they have the following features

	Alloy		
Properties	**1**	**2**	**3**
contains tin (in %)	60	40	20
contains zinc (in %)	10	40	50
contains lead (in%)	30	20	30
price (PLN per dag)	0,44	0,40	0,50

Formulate a linear problem that will find the cheapest recipe for the final product. Adjust the units (1 dag= 0,01 kg).

4. At home, you get the news: your lovely aunt sent you 6000PLN for you birthday. You can invest the money to the business run by your older brother: he wants 5000PLN and 400 hours of your work — this will bring the profit of 4500. However, your younger sister also has an offer for you: give her 4000PLN and work 500 hours in her small enterprise — the profit will be the same.

You feel like working 600 hours in the near future so you decide to divide you money and work between those offers. Which proportion will be optimal?

5. Consider the problem

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leqslant 3 \\ & 2x_1 + x_2 \leqslant 3 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Draw the corresponding polyhedron on the plane. Find the optimal solution (examining the picture).

6. Note that a polyhedron is a subset of the Euclidean space that *may* be defined by a finite number of linear inequalities/equations.

Is the set of $(x, y) \in \mathbb{R}^2$ satisfying $x^2 - 8x + 15 \leqslant 0$ and $y \geqslant 0$ a polyhedron?

What about the set of $(x, y) \in \mathbb{R}^2$ such that $x \cos\theta + y \sin\theta \leqslant 1$ for all $\theta \in [0, \pi/2]$?

7. Recall that the standard form of a linear problem is $\min c \cdot x = \sum_i c_i x_i$ subject to $Ax = b$ and $x_1, \dots, x_n \geqslant 0$, where A is some $m \times n$ matrix.

Find a standard problem that is equivalent to

$$\begin{aligned} &\text{maximize} && x_1 + 2x_2 - 3x_3 \\ &\text{subject to} && x_1 + x_2 - x_3 \leqslant 5 \\ &&& x_2 + x_3 \geqslant 4 \\ &&& x_1, x_2 \geqslant 0. \end{aligned}$$

HINT. Write $x_3 = x_3^+ - x_3^-$; replace every inequality by an equality with an additional nonnegative variable.

8. Sketch the convex hull of the following collection of points in $\mathbb{R}^2$

$$(0,0), (1,1), (-1,-1), (-2,2), (1,4), (0,3), (-1,1), (\frac{1}{2},4), (-1,2), (2,5).$$

Find those points which are vertices. Note that other points are convex combinations of at most three vertices.

9. Consider the polyhedron P given by

$$\begin{aligned} 2x_1 - x_2 &\geqslant -2 & \qquad x_1 + 2x_2 &\leqslant 8 \\ x_1 &\geqslant 0 & x_2 &\geqslant 0 \end{aligned}$$

Find (examining the picture of P) $\min x_2$, $\min(3x_1+2x_2)$, $\min(2x_1+4x_2)$, $\max(2x_1-2x_2)$, $\max(-3x_1 - 2x_2)$ for $(x_1, x_2) \in P$.

10. Let H be a hyperplane in $\mathbb{R}^n$, that is $H = \{x \in \mathbb{R}^n : a \cdot x = b\}$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Prove that if the set $C \subseteq \mathbb{R}^n \setminus H$ is convex then C is contained in one of the half-spaces $\{x \in \mathbb{R}^n : a \cdot x > b\}$, $\{x \in \mathbb{R}^n : a \cdot x < b\}$.

11. Caratheodory's theorem: *Let P be the convex hull of $a^1, \dots, a^m \in \mathbb{R}^n$. Then every $x \in P$ can be written as a convex combination $\sum_{i=1}^m \lambda_i a^i$, with at most $n+1$ non zero coefficients λ_i.*

This is illustrated by 8 above. To prove the theorem you may follow the idea given below.

(i) we know that every $v \in P$ is a convex combination $v = \sum_{i=1}^m \lambda_i a_i$. We shall check that if $m > n+1$ then one can reduce the number of pieces by one

(ii) Since $m - 1 > n$, the vectors $a^1 - a^m, a^2 - a^m, \dots, a^{m-1} - a^m$ are not independent so there are c_i such that $\sum_{i=1}^m c_i a^i = 0$ and $\sum_{i=1}^m c_i = 0$.

(iii) We can write $v = \sum_1^m (\lambda_i - tc_i)a^i$ where $t \in \mathbb{R}$. Choose a suitable value of $t \geqslant 0$.

12. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda y)) \leqslant \lambda f(x) + (1 - \lambda) f(y),$$

for $x, y \in \mathbb{R}^n$ and $0 \leqslant \lambda \leqslant 1$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if in the formula above $\leqslant$ is replaced by $\geqslant$.

Prove that $f : \mathbb{R}^n \to \mathbb{R}$ which is concave and convex at the same time is affine, that is $f(x) = a \cdot x + b$ for some $a \in \mathbb{R}^n, b \in \mathbb{R}$.

HINT: Show that $F(x) = f(x) - f(0)$ is linear.

13. Jensen's inequality: *If the set $C \subseteq \mathbb{R}^n$ is convex and $f : C \to \mathbb{R}$ is a convext function then*

$$f(\sum_{i=1}^k \alpha_i x^i) \leqslant \sum_{i=1}^k \alpha_i f(x^i),$$

for every convex combination of $x^i \in C$.

Check it by induction.