The simplex method

1. Consider the following problem: minimize $x_1 + x_2 + 2x_3 + 3x_4$ subject to

$$\begin{array}{rcrcrcrcl} 3x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2, \\ 2x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & 3, \\ & & & & & x_1, x_2, x_3, x_4 & & \geqslant & 0. \end{array}$$

Find the basic solution taking columns A_3, A_4 as a basis.

Find the direction $d = (0, 1, d_3, d_4)$ and calculate the reduced costs.

Find θ^* such that $y = x + \theta^* d$ is another vertex and check whether y is optimal.

2. Consider the following problem: minimize $3x_1 + 2x_2$ subject to

$$\begin{array}{rcrcrcrcrcl} x_1 & & & + & x_3 & & & & & = & 4, \\ x_1 & + & 3x_2 & & & - & x_4 & & & = & 15, \\ 2x_1 & + & x_2 & & & & & - & x_5 & = & 10 \\ & & & & & & x_1, x_2, x_3, x_4, x_5 & & & \geqslant & 0. \end{array}$$

Find the basic solution taking columns A_1, A_2, A_3 as a basis and check whether this solution is optimal.

3. Consider now: maximize $x_1 + 2x_2$, subject to

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & \leqslant & 8, \\ x_1 & + & x_2 & \leqslant & 4, \\ & & x_1, x_2 & \geqslant & 0. \end{array}$$

Reduce the problem to the standard form and find its basic feasible solutions.

4. Use (some of) the examples above to exercise how to write the full tableau for a given problem and how to iterate the algorithm.

5. Suppose that we have got the following tableau

	-3	-2	9	1	6	0	0	0
x_5	1	2	-8	-1	4	1	0	0
x_6	2	1	-1	-1	3	0	1	0
x_7	3	-2	0	1	0	0	0	1

Perform the next iteration and derive conclusions.

6. Suppose that we have got the following tableau

	-3	c_1	c_2	2	4	0	0	0
x_5	1	u_1	-8	-1	4	1	0	0
x_6	2	u_2	-1	-1	3	0	1	0
x_7	2	u_3	0	1	0	0	0	1

What are the conclusions in the following cases

(a) $c_1, c_2 \geqslant 0$;

(b) $c_1 < 0$, $u_1 = 1$, $u_2 = 2$ $u_3 = 3$;

(c) $c_1 < 0$, $u_2 = 2$, $c_2 < 0$;

(d) $c_1 < 0$, $u_2 = 2$, $c_2 > 0$;

(d) $c_1 < 0$ and $u_1, u_2, u_3 < 0$;

(e) $c_2 < 0$?

7. Explain why (while considering the tableau)

(a) if the minimal value of $\theta^* = \frac{x_{B(l)}}{u_l}$, is attained for two different indices l then the next solution is degenerate;

(b) if the pivoting column has only negative entries then the problem in question does not have optimal solutions;

(c) if we have non-degenerate basic feasible solution and some reduced cost is negative then the solution is not optimal.

8. Find examples showing that

(a) a given (degenerate) solution may be optimal even if some reduced costs are negative;

(b) there may bo no advantage of choosing a pivoting column with the least reduced cost.

9. How we reduce the costs?

We consider $\min c \cdot x$, subject to $Ax = b,\ x \geqslant 0$, where A is a $m \times n$; we assume that those m equations are linearly independent (by the way, why?).

Let x be the basic feasible solution corresponding to columns $A_{B(1)}, \ldots, A_{B(m)}$. Let $R_1, R_2, \ldots, R_m$ denote the rows of the matrix A.

(a) Show that there are unique $\lambda_1, \ldots, \lambda_m$ such that the vector

$$c - \sum_{i=1}^{m} \lambda_i R_i$$

has zero coordinates $B(1), \ldots, B(m)$.

(b) Show that the rows of $B^{-1}A$ are linear combinations of those of A.

(c) Show that the vector $\bar{c}$ of reduced costs is of the form

$$c - \sum_{i=1}^{m} \mu_i R_i.$$

(d) Finally, $\mu_i = \lambda_i$.

This explains why we use row operations to calculate the reduced costs.

10. A reflection on $-c \cdot x$ in the tableau.

Continuing the previous item, let $\overline{c}$ be the vector of reduced costs; suppose $\overline{c}_j < 0$ and the minimal valu of θ is

$$\theta^* = \frac{x_{B(l)}}{u_l};$$

here u_l is the lth coordinate of the vector $u = -d_B = B^{-1}A_j$. Then $y = x - \theta^* u$ becomes the next BFS connected with the new basis $B(1), \ldots, B(l-1), j, B(l+1), \ldots, B(m)$.

Check that we can calculate the new vector of reduced costs by the rule

$$\text{new reduced costs} = \text{ old reduceed costs } - \lambda \times \text{ the } i\text{th row of } B^{-1}A,$$

where λ is chosen to change $\overline{c}_j$ to zero.

Note that

$$-c \cdot y = -c \cdot x - \lambda x_{B(l)},$$

to see why we write $-c \cdot x$ in the tableau.

11. Consider a general problem (P) in its standard form $\min c \cdot x$ subject to $Ax = b$, $x \geqslant 0$ where A is a $m \times n$ matrix. Suppose that (P) has some feasible solutions.

Note that we may have $\binom{n}{m}$ basic solution so to find the initial basic feasible solution may be troublesome.

Note also that we can assume that $b \geqslant 0$ and consider an auxiliary problem (AP) $\min(y_1 + \ldots + y_m)$ subject to $Ax + y = b$ and $x \geqslant 0$, $y \geqslant 0$.

Then $(x, y) = (0, b)$ is a basic feasible solution of (AP). Let (x^*, y^*) be an optimal solution of (AP). Check that we have then $y^* = 0$ and x^* is a basic feasible solution of (P).