1. Using the general theory of linear problems, describe a simple way of finding $\min \sum_{i=1}^{n} c_i x_i$ subject to $\sum_{i=1}^{n} p_i x_i = b \in \mathbb{R}$, $x_1, \ldots, x_n \geqslant 0$.

2. Consider a problem $\min c \cdot x$ subject to $Ax \geqslant b$ and $x \geqslant 0$, where A is an $m \times n$, matrix, $c, x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

We standardize it using $y \in \mathbb{R}^m$: $\min c \cdot x$ subject to $Ax - y = b$ and $x, y \geqslant 0$. Check that if (x^*, y^*) is an optimal solution of that standard problem then x^* is optimal for the starting problem. Write the standard problem as

$$\mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix} = b.$$

What its the rank of $\mathcal{A}$? Check that if (x, y) is a basic solution of the extended problem then X is a basic solution of the original one.

3. Back to Caratheodory's theorem (see 11/List 1). Every point of a poyhedron P in $\mathbb{R}^n$ is a convex combination of at most $n+1$ of its extreme points.

We know that $x \in P$ is a convex combination of some extreme points $e^1, \ldots, e^k$ of P. To prove the theorem consider the polyhedron $L \subseteq \mathbb{R}^k$ of those $\lambda = (\lambda_1, \ldots, \lambda_k)$ that represent x as a convex combination of e^j. Then consider a basic feasible solution of L.

4. Lexicographic rule for the tableau. Recall that if all the BFS are non-degenerate then we lower the cost at each iteration of the tableau and we are bound to finish after a finite number of steps. However, if there are degenerate BFS then unlucky choice of pivoting elements may produce a loop in the algorithm. This can be avoided as follows.

For $x \neq y \in \mathbb{R}^n$ write $x <_{lex} y$ if $x_i < y_i$ for the first i such that $x_i \neq y_i$. In particular, if $\mathbf{0} <_{lex} x$ then we say that x is lexicographically positive (that is the first nonzero coordinate of x is > 0. Note that $<_{lex}$ is a linear order on $\mathbb{R}^n$.

Consider a starting tableau for the simplex method. Note that by changing enumeration of the variables, we can assume that all the rows, except for the zero row, are lexicographically greater than the zero vector. Then we apply the following rules while iterating:

(a) We choose a pivoting column arbitrarily (as long as the reduced cost is negative).

(b) If the pivoting element is uniquely defined then we proceed; otherwise, $x_{B(i)}/u_i$ is minimal for various i's. To decide which one we choose we divide the i-th row by u_i and choose i such that the result is lexicographically smallest.

Check that in such a case

(i) Every row (except for the zero row) remains lexicographically positive;

(ii) The zero row becomes lexicographically bigger after each operation (so the will be no loops!).

Duality

5. Find solutions of the following two (separate) problems using duality (e.g. write in each case the dual problem and guess vectors giving the same values of the cost functions).

$$\begin{array}{l} \min x_1 + x_2 \\ \begin{array}{rcrcr} x_1 & + & 2x_2 & \geqslant & 2 \\ x_1 & & & \geqslant & 1 \\ x_1 & - & 3x_2 & \geqslant & -1 \\ & & x_1, x_2 & \geqslant & 0 \end{array} \end{array} \qquad \begin{array}{l} \max x_1 + x_2 \\ \begin{array}{rcrcr} x_1 & + & 2x_2 & \leqslant & 2 \\ x_1 & & & \leqslant & 1 \\ x_1 & - & 3x_2 & \leqslant & -1 \end{array} \end{array}$$

6. Find a standard problem (P) to which the following is dual (DP):

$$\max -2x_1 - 3x_2$$

$$\begin{array}{rcrcr} x_1 & & & \geqslant & 1 \\ x_1 & + & 2x_2 & \leqslant & 3 \\ -x_1 & + & x_2 & \leqslant & 2 \end{array}$$

Note that (P) has no feasible solutions at all. What does this say about (DP)?

7. An example from the lecture: check that $(1, 0, 1)$ is an optimal solution of

$$\begin{array}{l} \min 13x_1 + 10x_2 + 6x_3 \\ \begin{array}{rcrcrcr} 5x_1 & + & 3x_2 & + & 3x_3 & = & 8 \\ 3x_1 & + & x_2 & + & & = & 3 \\ & & & & x_1, x_2, x_3 & \geqslant & 0 \end{array} \end{array}$$

8. Another example: find an optimal solution of the problem $\min(x_1 + 2x_2 + \ldots + nx_n)$ subject to the constraints $\sum_{i=1}^{k} x_i \geqslant k$ for $k = 1, 2, \ldots, n$ and $x_1, \ldots, x_n \geqslant 0$.

9. A 24/7 restaurant needs to have the staff consisting of

Workers	Hours
2	03 - 07
10	07 - 11
14	11 - 15
8	15 - 19
10	19 - 23
3	23 - 03

Everyone works 8 hours. Formulate the problem minimizing the number of employees; find the dual problem and check that the vector $(0, 14, 0, 8, 2, 2)$ (deciding how many should start their shift at 3,7, etc.) is optimal.

10. Consider (P)

$$\min c \cdot x \text{ subject to } Ax = b, x \geqslant 0,$$

where $x \in \mathbb{R}^n$, A is an $m \times n$ matrix etc. Write the dual problem (DP) and its standard form (SDP). Then find the dual of (SDP).

11. Strong duality says that if we have some linear problem (P), say $\min c \cdot x$ for $x \in A$, and we form its dual (DP) then, in case both problems are feasible (have some solutions) then both have optimal solutions, in particular $c \cdot x$ is bounded from below on A. Reading carefully the proofs, one can conclude the reverse: that if (P) is feasible but (DP) is infeasible then $c{\cdot}x$ is unbounded from below on A. This has some theoretical consequences — one example is given below.

Consider a stochastic matrix A of size $n \times n$; this means that its entries are nonnegative and sum of every row equals 1. There is a theorem (in the theory of Markov chains) that there is a vector $p = (p_1, \ldots, p_n) \geqslant 0$ such that $\sum_j p_j = 1$ and $A^T p = p$.

You can prove it using duality: consider the standard problem $\min(-x_1 - \ldots - x_n)$ for $(A^T - I)x = 0$, $x \geqslant 0$, where I is the unit matrix. Write its dual and check that it is not feasible so, by the above, the original problem is unbounded.

12. Using the above fact on duality one can also prove the following:

If the system of inequalities $a^i{\cdot}x \leqslant c_i$, $i = 1, \ldots, k$, where $x \in \mathbb{R}^n$ and $k > n$ is inconsistent then there are $n+1$ inequalities among them which are already inconsistent.

From this we get Helly's theorem: if $P_1, \ldots, P_k$ are polyhedra in $\mathbb{R}^n$ and every $n+1$ of them have nonempty intersection then $\bigcap_{i=1}^k P_i \neq \emptyset$.

13. Dual simplex method — an example. Suppose that we have got a basic solution which is infeasible (for a given standard linear problem) but the reduced costs turn out to be all nonnegative. An observation from the lecture says that we then can define a feasible solution of the dual problem (but we keep thinking about the original one). An example in a tableau:

$$\begin{array}{c|c|ccccccc|}
\cline{2-9}
 & -3 & 1 & 2 & 2 & 4 & 0 & 0 & 0 \\
\cline{2-9}
x_5 & a & v_1 & v_2 & v_3 & v_4 & 1 & 0 & 0 \\
x_6 & 1 & 3 & -1 & -1 & 3 & 0 & 1 & 0 \\
x_7 & 2 & 4 & 0 & 1 & 0 & 0 & 0 & 1 \\
\cline{2-9}
\end{array}$$

Here $a < 0$ (otherwise, we have a BFS which is optimal). We can perform analogous row operations, this time we fight for nonnegative entries in the zero column and keep the vector of reduced costs nonnegative. The first move is to multiply the first row by -1. If v_1, v_2, v_4, v_4 were nonnegative then we conclude that our problem is infeasible (because the dual problem is unbounded). If, say $v_1, v_2 < 0$, $v_3, v_4 > 0$ then examine which is smaller $c_1/|v_1|$ or $c_2/|v_2|$ to choose the pivoting element. As before apply row operations to get the cost 0 in the pivoting column and the unit vector below it (by our choice, the reduced costs remain nonnegative).

14. Apply the dual tableau to the following problems. Here, after standardization, we get an obvious basic solution which is not feasible but has nonnegative reduced costs.

$$\begin{array}{lcrcl}
\multicolumn{5}{l}{\min x_1 + x_2} \\
x_1 & + & 2x_2 & \geqslant & 2 \\
x_1 & & & \geqslant & 1 \\
 & & x_1, x_2 & \geqslant & 0
\end{array}
\qquad
\begin{array}{lcrcrcr}
\multicolumn{7}{l}{\min : x_1 + 3x_2 + x_3} \\
2x_1 & - & 5x_2 & + & x_3 & \leqslant & -5 \\
2x_1 & - & x_2 & + & 2x_3 & \leqslant & 4 \\
 & & & & x_1, x_2, x_3 & \geqslant & 0
\end{array}$$