1. Check that the identity $I : \ell_1 \to \ell_2$, $Ix = x$ is a bounded operator but its inverse is not continuous. What about the open mapping theorem?

2. Write a formula for an isomorphism $T : c \to c_0$ (c is the space of all converging sequences).

3. Prove that c is not isometric to c_0.

HINT: Think of extreme points of the unit balls in c and in c_0.

4. Let K be the one-point compactification of $\mathbb{N}$, that is $K = \mathbb{N} \cup \{\infty\}$ and the basic neighbourhoods of ∞ are of the form $\{\infty\} \cup (\mathbb{N} \setminus F)$ for finite F. Note that $C(K)$ and c are isometric.

5. Check that $L_1[0,1]$ and $L_1(\mathbb{R})$ are isometric.

6. Prove that a subset of ℓ_1 is weakly compact if and only if it is compact in the norm topology.

HINT: By the Eberlein-Smulyan theorem, weak compactness is equivalent to weak sequential compactness (every sequence has a converging subsequence).

7. Prove that for a sequence of measurable $f_n\colon [0,1] \to [0,1]$, $f_n \to 0$ weakly in $L_1[0,1]$ if and only if $\int_a^b f_n \,\mathrm{d}\lambda \to 0$ for every $0 \leqslant a < b \leqslant 1$.

8. Let $f_n(t) = \sin(2\pi t n)$. Check that f_n converge weakly to 0 in $L_1[0,1]$ but not in norm.

9. Let λ be a probability measure on a σ-algebra Σ of subsets of some T; let $\mathcal{A} \subseteq \Sigma$ be a smaller σ-algebra; write $\mu = \lambda|\mathcal{A}$. Prove that $L_1(T, \mathcal{A}, \mu)$ is complemented in $L_1(T, \Sigma, \lambda)$.

HINT: Think of the conditional expectation.

10. Note that $C[0,1]$ may be treated as a subspace of $L_\infty[0,1]$. Prove that $C[0,1]$ is not complemented in $L_\infty[0,1]$ as follows.

Pick norm-one functions $g_n \in C[0,1]$ such that $g_n \cdot g_k = 0$ for $n \neq k$. Then $f_J = \sum_{n\in J} g_n \in L_\infty[0,1] \setminus C[0,1]$ for every infinite $J \subseteq \mathbb{N}$. Think of the proof of Phillip's theorem about $c_0 \subseteq \ell_\infty$. Hint for the final step: for every measurable $B \subseteq [0,1]$ with $\lambda(B) > 0$ there is a nonempty interval (p,q) with rational endpoints such that $\lambda(B \cap (p,q)) > 3/4(q-p)$.

EBERLEIN COMPACTA (1)

11. A compact space K is Eberlein compact if it is homeomorphic to a weakly compact subset of some Banach space. Eberlein and Smulyan say, in particular, that every Eberlein compact has the Frechet property (the closure inside it is determined by limits of converging sequences).

Prove that every metrizable compactum is Eberlein compact.

12. Check that for *every* compact space K, $K \hookrightarrow C(K)^*$. Consequently, *weak** compacta need not be Eberlein compact.

13. Suppose that a Banach space X is the closed linear span of its weakly compact subset K. Prove that, in such a case, $(B_{X^*}, weak^*)$ is Eberlein compact.

HINT: Note first that $B_{X^*} \ni x^* \to x^*|K \in C(K)$ is one-to-one.