Boolean algebras and Stone spaces

Recall that for a Boolean algbera $\mathfrak{A}$, its Stone space $\mathrm{ult}(\mathfrak{A})$ consists of all utrafilters on $\mathfrak{A}$ and the sets $\widehat{a} = \{\mathcal{U} : a \in \mathcal{U}\}$, where $a \in \mathfrak{A}$, form a base for the compact Hausdorff topology on $\mathrm{ult}(\mathfrak{A})$.

1. Show that in the algebra $\mathfrak{A} = \mathcal{P}(\omega)/\mathrm{fin}$

(i) there is $\mathfrak{c}$-many pairwise disjoint nonzero elements;

(ii) if $a_n \neq 0$ and $a_1 \geqslant a_2 \geqslant \ldots$ then there is $b \neq 0$ such that $b \leqslant a_n$ for every n.

(iii) if $a_1 \geqslant a_2 \geqslant \ldots \geqslant b_2 \geqslant b_1$ then there is x such that $b_n \leqslant x \leqslant a_n$ for every n.

2. Recall that $\mathrm{ult}(\mathcal{P}(\omega))$ is denoted by $\beta\omega$. Note that we may think that $\omega \subseteq \beta\omega$ since every principal ultrafilter $\{A \subseteq \omega : n \in A\}$ may be identified with n.

Check that a nonprincipial ultrafilter $\mathcal{U}$ may be seen as an ultrafilter on $\mathcal{P}(\omega)/\mathrm{fin}$ so $\mathrm{ult}(\mathcal{P}(\omega)/\mathrm{fin})$ is then $\beta\omega \setminus \omega$.

3. Check that the compact space $\beta\omega$ contains no nontrivial converging sequence.

4. If you like Banach algebras: note that every multiplicative functional on ℓ_∞ corresponds to some ultrafilter on ω.

5. Let $\mathfrak{B} = Bor[0,1]/\mathcal{N}$ be the measure algebra; here $\mathcal{N} = \{A \in Bor : \lambda(A) = 0\}$. Prove that $\mathfrak{B}$ is complete (while $Bor[0,1]$ is only σ-complete).

Hint: For any $\{b_t : t \in T\} \subseteq \mathfrak{B}$, $b_t = B_t/\mathcal{N}$, there is a countable $T_0 \subseteq T$ such that $\bigcup_{t \in T_0} B_t$ has the maximal possible measure.

6. Prove that the Stone space of the measure algebra is not separable

Extremally disconnected spaces

7. A topological space K is extremally disconnected if the set $\overline{U}$ is open for every open $U \subseteq K$ (no misprints here:-).

Check that K is extremally disconenected if and only if $\overline{U} \cap \overline{V} = \emptyset$ for every disjoint open $U, V \subseteq K$.

8. Prove that the Stone space $\mathrm{ult}(\mathfrak{A})$ is extremally disconenected if and only if the algebra $\mathfrak{A}$ is complete.

Hint: If $U \subseteq \mathrm{ult}(\mathfrak{A})$ is open then $U = \bigcup_t \widehat{a_t}$ for some $a_t \in \mathfrak{A}$; check that if $a = \bigvee_t a_t$ exists in $\mathfrak{A}$ then $\overline{U} = \widehat{a}$.

Extending functions

9. Prove that for any ultrafilter $\mathcal{U}$ on ω and any sequence x_n in a compact space K there is the unique limit $x = \lim_{n \to \mathcal{U}} x_n \in K$ such that $\{n \in \omega : x_n \in V\} \in \mathcal{U}$ for every open neighbourhood $V \ni x$.

Hint: Otherwise, every $x \in K$ has a *bad* neighbourhood V_x and...

10. Check that for any $f : \omega \to K$, if K is compact then $f^\beta(\mathcal{U}) = \lim_{n\to\mathcal{U}} f(n)$ defines an extension of f to a continuous function $f^\beta : \beta\omega \to K$.

11. In the case of bounded $f, g : \omega \to \mathbb{R}$, check that

$$\lim_{n\to\mathcal{U}} (f(n) + g(n)) = \lim_{n\to\mathcal{U}} f(n) + \lim_{n\to\mathcal{U}} g(n);$$

in other words $(f + g)^\beta = f^\beta + g^\beta$.

12. Note that it follows from (10) that every separable compact space is a continuous image of $\beta\omega$. In particular, $|\beta\omega| = 2^{\mathfrak{c}}$.

HINT: The cube $\{0, 1\}^{\mathbb{R}}$ is separable: think of characteristic functions of finite unions of intervals with rational endpoints.

13. A remark: the notation β refers to the so called maximal compactification. For instance, if $\gamma\omega$ is any compactification of ω (a compact space containing ω as a dense subset) then there is a continuous surjection $\beta\omega \to \gamma\omega$ which is constant on ω (see above).