Projective resolution of compacta

1. A set U in a topological space is regular open if $U = \text{int}(\overline{U})$.

Check that the family $RO(T)$ of such sets in a topological space T forms a Boolean algebra with the operations $U \wedge V = U \cap V$, $U \vee U = \text{int}(\overline{U \cup V})$, $U^c = \text{int}(T \setminus U)$.

2. Prove that the algebra $RO(T)$ is always complete.

3. A continuous surjection $g : K \to L$ (between compact spaces) is irreducible if $g[F] \neq L$ for every proper closed subset F of K.

Check that irreducibility is equivalent to saying that $\text{int}(g[U]) \neq \emptyset$ for every nonempty open $U \subseteq K$. Prove, using Zorn's lemma, that for every continuous surjection $g : K \to L$ there is closed $K_0 \subseteq K$ such that $g|K_0 : K_0 \to L$ is an irreducible surjection.

4. If K is compact; the projective resolution of K (or Gleason's space) $G(K)$ is the Stone space of $RO(K)$. It follows that $G(K)$ is extremally disconnected space.

Prove that there is a canonical irreducible continuous surjection $\pi : G(K) \to K$.

Hint: Define $\pi(p)$ to be the only point in the intersection of all $\overline{U}$ for U belonging to the ultrafilter p.

Remark: $G(K)$ is unique up to homeomorphism.

Eberlein compacta (2) and beyond

5. For any index set Γ we put

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : |x(\gamma)| \geqslant \varepsilon\}| < \omega \text{ for every } \varepsilon > 0\};$$

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leqslant \omega\}.$$

Check that $c_0(\Gamma)$ is a Banach space in sup-norm — a generalized version of c_0. Moreover, $c_0(\gamma)^*$ is $\ell_1(\Gamma)$ (which is defined in the usual manner).

6. Note that a weakly compact subset of a separable Banach space embeds into c_0.

By a deep theorem due to Amir and Lindenstrauss, every weakly compact set in any Banach space embeds into some $c_0(\Gamma)$ equipped with the topology of pointwise convergence inherited from $\mathbb{R}^\Gamma$. In other words, we may define Eberlein compacta as compact spaces embeddable into some $c_0(\Gamma)$.

Check that, with such a definition, an Eberlein compact is separable iff it is metrizable iff it is *ccc*.

7. Those compact spaces that can be embedded into some $\Sigma(\mathbb{R}^\Gamma)$ are called Corson compacta. Note that $c_0(\Gamma) \subseteq \Sigma(\mathbb{R}^\Gamma)$ so every Eberlein compact space is Corson compact.

Prove that every Corson compact space is sequentially compact.

8. Prove that every Corson compact space K has the Frechet property (that is, if $x \in \overline{A}$ for $A \subseteq K$ then $x = \lim_n a_n$ for some $a_n \in A$.

HINT: Let the support of x be $\{\gamma : x(\gamma) \neq 0\} = \{\gamma_1, \gamma_2, \ldots\}$. Approximate x by $a_1 \in A$ on the first coordinate of the support; then approximate x by a_2 on the two coordinates of the support of x and two coordinates of $a_1 \ldots$

9. Prove that every Corson compact space K contains a point x with a countable local base.

HINT: There is a countable base at a point x in a compact space iff $\{x\}$ is G_δ. Consider $x \in K \subseteq \Sigma(\mathbb{R}^\Gamma)$ which is maximal with respect to pointwise order.

10. Note that every separable Corson compact space is metrizable. However, the sentence 'every *ccc* Corson compact is metrizable' is undecidable; if you know Martin's axiom then check that $\mathrm{MA}(\omega_1)$ implies such a statement. On the other hand, CH or some weaker statements imply the opposite — see below.

11. Assume that $[0,1] = \bigcup_{\xi<\omega_1} N_\xi$ where every N_ξ is Lebesgue null. Pick a closed

$$F_\xi \subseteq [0,1] \setminus \bigcup_{\alpha<\xi} N_\alpha$$

such that $\lambda(F_\xi) > 0$. Note that the family $\mathcal{F} = \{F_\xi : \xi < \omega_1\}$ has the property that every its subfamily with nonempty intersection is countable.

Let $\mathcal{A}$ be a family of $A \subseteq \omega_1$ such that for every finite $I \subseteq A$ we have $\lambda(\bigcap_{\xi \in I} F_\xi) > 0$. Note that $K = \{\chi_A : A \in \mathcal{A}\}$ is a compact space.

Prove that K is a nonmetrizable Corson compact space satisfying *ccc*.