Suplement on ℓ_∞/c_0

1. Note that there is a one-to-one correspondence between nonprincipal ultrafilters on $\mathcal{P}(\omega)$ and ultrafilters on $\mathcal{P}(\omega)/\mathrm{fin}$. Then note that $\mathrm{ult}(\mathcal{P}(\omega)/\mathrm{fin}$ may be identified with $\beta\omega \setminus \omega$.

2. Recall that ℓ_∞ is isometric to $C(\beta\omega)$. Check that ℓ_∞/c_0 is isometric to $C(\beta\omega\setminus\omega)$. If $f \in \ell_\infty$ then we can define the required isometry by $T(f/c_0) = f^\beta|(\beta\omega \setminus \omega)$ (here f^β is the unique extension of f to a continuous function).

Suplement on separability of $C(K)$

3. If K is metrizable and compact then the Banach space $C(K)$ is separable: We can assume that $K \subseteq [0,1]^\omega$ so it remains to check that $C[0,1]^\omega$ is separable. Let $\mathcal{F}$ be a family of all projections $\pi_n : [0,1]^\omega \to [0,1]$ and let $\mathcal{F}'$ be the family of all finite products of functions from $\mathcal{F}$. Check that all the linear combinations of functions from $\mathcal{F}'$ form the equired countable dense set.

Around the Banach-Stone theorem

4. The theorem says that of $C(K)$ and $C(L)$ are **isometric** then K and L are homeomorphic. This can be proved as follows:

(a) Every extreme point in the dual ball $B_{C(K)^*}$ of signed measures of norm $\leqslant 1$ is of the form $\pm\delta_x$ for some $x \in K$.

(b) if $T : C(K) \to C(L)$ is an isometry then the dual operator $T^* : C(L)^* \to C(K)^*$ is an isometry so it sends extreme points of the ball to extreme points.

5. The Gelfand-Kolmogorov theorem says that if there is an isomorphism $T : C(K) \to C(L)$ such that $T(f \cdot g) = T(f) \cdot T(g)$ for $f, g \in C(K)$ then K and L are homeomorphic. This may be proved as follows:

(a) Let $\mathcal{I} = \{f \in C(K) : f(x_0) = 0\}$ where $x_0 \in K$. Then $\mathcal{I}$ satisfies $f, g \in \mathcal{I} \Rightarrow f + g \in \mathcal{I}$ and $f \in \mathcal{I}, g \in C(K) \Rightarrow f \cdot g \in \mathcal{I}$; we say that $\mathcal{I}$ is an ideal in the ring $C(K)$.

(b) Every ideal in $C(K)$ which is maximal among all proper ideals is of the form desrcibed above. Hint: if we suppose that there is $f_x \in \mathcal{I}$, $f_x(x) \neq 0$ for every $x \in K$ then, using compactness, we can show that $1_K \in \mathcal{I}$ so $\mathcal{I} = C(K)$.

(c) Now every T preserving multiplication sends maximal ideals in $C(K)$ to maximal ideals in $C(L)$.

Extension operators

6. For closed $F \subseteq K$, a bounded operator $T : C(F) \to C(K)$ is an extension operator if $Tg|F = g$ for every $g \in C(F)$. Note that if such an operator exists then $C(K) \simeq C(F) \oplus X$ where X is 2-complemented in $C(K)$. How to define X?

7. Prove that if K is a separable compact space and $F \subseteq K$ is its closed subspace that does not satisfy *ccc* then there is no extension operator $C(F) \to C(K)$.

In particular, there is no extension operator $C(\beta\omega \setminus \omega) \to C(\beta\omega)$.

AROUND MILJUTIN'S THEOREM

8. Check that in any Banach space X, every two hyperplanes (= subspaces of codimension 1) are isomorphic.

9. Note that for every infinite metrizable K, $C(K)$ comtains a copy of c_0 which is then complemented by Sobczyk's theorem. Conclude that $C(K+1)$ is isomorphic to $C(K)$; here $K+1$ denotes K with one isolated point added.

10. The above implies that for a compact metrizable K the space $C(K)$ si isomorphic to any its hyperplane.

11. Prove directly that $C[0,1] \simeq C[0,1] \oplus C[0,1]$.

HINT: $C[0,1] \simeq \{f \in C[0,1] : f(1/2) = 0\}$.

12. Try to prove that if $\theta : 2^\omega \to [0,1]$ is the canonical surjection, that is

$$\theta(x) = \sum_{n=0}^{\infty} x(n)/2^{n+1},$$

then the subspace $\{g \circ \theta : g \in C[0,1]\}$ of $C(2^\omega)$ is not complemented.