(1) Consider $f \in L_1[0,1]$ and write

$$\Theta(x) = \limsup_{\delta \to 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f$$

Assume that $f \geq 0$

(2) Using simple functions

$$0 \leq s_1 \leq s_2 \leq \dots \to f$$

it is easy to show that

$$\liminf_{\delta \to 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f \geq f(x)$$

almost everywhere. Hence it remains to show that $\Theta(x) \leq f(x)$ almost everywhere.

(3) Recall that, since $f \in L_1$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A \in \text{Borel} \quad \lambda(A) < \delta \Rightarrow \int_A f \, d\lambda < \varepsilon$$

(4) To check $\Theta(x) \leq f(x)$ take any $p, q \in \mathbb{Q}$ $p < q$ and consider

$$A_{p,q} = \{x : \Theta(x) > q,\ f(x) < p\}$$

and prove $\lambda(A_{p,q}) = 0$

(5) Assume $\lambda(A_{p,q}) > 0$, take open $V \supseteq A_{p,q}$ such that $\int_{V \setminus A_{p,q}} f$ is <u>veeery</u> small.

Apply Vitali to intervals $I \subseteq V$ such that

$$\frac{1}{\lambda(I)} \int_I f > q$$

to get a contradiction.