...cólera[1]

21. Appendix on the Riesz representation theorem

For any compact space K we write $P(K)$ for the space of all probability Borel measures on K that are regular (with respect to closed sets). Recall that, unlike in the metrizable case, regularity is no longer automatic. The Riesz theorem says the following:

Theorem 21.1. *Given a compact space K, if $\varphi \in C(K)^*$ is a positive functional on $C(K)$ such that $\varphi(\chi_K) = 1$ then there is a unique $\mu \in P(K)$ that represents φ by integration.*

As before $P(K)$ is given the *weak**-topology, the one generated by all the mappings $\mu \mapsto \int_K g \, \mathrm{d}\mu$, $g \in C(K)$.

Theorem 21.2. *The space $P(K)$ is compact.*

Below we comment on possible proofs of these results. Recall that K is zero-dimensional if the family clop(K), of all closed and open sets, is a topological base.

21.1. **Thm. 21.1 implies Thm. 21.2.** The Banach-Alaoglu theorem says that for any Banach space X, the dual unit ball B_{X^*} is compact in the *weak** topology. This is fairly standard: the mapping $B_{X^*} \ni x^* \mapsto (x^*(x))_{x \in B_X}$ is an embedding into the Tikhonov cube $[-1,1]^{B_X}$ and remains to note that the image is closed. In turn, $P(K)$ is a closed subset of $B_{C(K)^*}$ so it is compact as well.

21.2. **Thm. 21.2 implies Thm. 21.1.** Once we know that $P(K)$ is compact, it is enough to check that $P(K)$ is *weak** dense in the set of functionals as in 21.1.

Consider a norm-one positive functional φ on $C(K)$ and a finite collection of nonnegative functions $f_i \in C(K)$, $i \le n$. We claim that there is a measure $\mu \in P(K)$ supported by a finite set such that $\varphi(f_i) = \int_K f_i \, \mathrm{d}\mu$ for every $i \le n$. **Exercise:-)** (Hint: think of Hahn-Banach in finite dimensional spaces.)

21.3. **Proof of Thm. 21.1 in the zero-dimensional case.** We define $\mu(A) = \varphi(\chi_A)$ for $A \in$ clop(K). Then μ is a finitely additive measure on the algebra clop(K) and we can apply the procedure of regular extensions for the lattice $\mathcal{K} =$ clop(K) and $\mathcal{L}$ equal to the lattice of all closed subsets of K. Hence μ extends to a regular Borel measure on K. The uniqueness follows from the fact that for every closed F and open $U \supseteq F$ there is $A \in$ clop(K) such that $F \subseteq A \subseteq U$.

[1] now of a political flavour

21.4. **Proof of Thm. 21.2 in the zero-dimensional case.** First note that

$$P(K) \ni \mu \mapsto \big(\mu(A)\big)_{A\in \mathrm{clop(K)}},$$

is an injective continuous mapping so it is a homeomorphism onto its image (consisting of all finitely additive probability measures on clop((K). In turn, it is not difficult to check that such a space is closed subset of $[0,1]^{\mathrm{clop(K)}}$.

21.5. **General topological fact:** *Every compact space K is a continuous image of a compact zero-dimensional space K_0.*

Indeed, we may assume that $K \subseteq [0,1]^\kappa$ for some κ. Since there is a continuous surjection $g : 2^\omega \to [0,1]$, there is a continuous surjection $\theta : \left(2^\omega\right)^\kappa \to [0,1]^\kappa$. Now $K_0 = \theta^{-1}[K]$ is as required.

Exercise: Why there are compact spaces that are not a continuous image of Cantor cubes of the form 2^κ?

21.6. **Theorem:** *Suppose that $\theta : K \to L$ is a continuous surjection between compact spaces. Then*

$$P(K) \ni \mu \mapsto \theta[\mu] \in P(L)$$

is a continuous surjection too.

The continuity of such a mapping follows from the formula

$$\int_L g \; \mathrm{d}\theta[\mu] = \int_K g\circ\theta \; \mathrm{d}\mu.$$

Take any $\nu \in P(L)$; then we can define μ_0 on the σ-algebra $\Sigma = \{\theta^{-1}(B) : B \in Bor(L)\}$ by the formula $\mu_0(\theta^{-1}[B] = \nu(B)$. It remains to check that μ_0 extends to $\mu \in P(K)$.

For this we apply the regular extension procedure: μ_0 is inner-regular with respect to the lattice $\mathcal{K} = \{\theta^{-1}[F] : F = \overline{F} \subseteq L\}$ so it extends to a regular Borel measure.

21.7. **Proof of Thm. 21.2:** Apply 21.6 , 21.5 and 22.4.

21.8. **Proof of Thm. 21.1:** Apply 21.5 and 22.3 and the Hahn-Banach theorem.

In fact, here we need to know that a positive linear functional on a subspace of a Banach lattice admits a positive extension to. a linear functional on the whole space.

22. The Haar measure on a compact group

We reproduce here a clever construction due to von Neumann. This is discussed in a monograph by Diestel and Spalsbury[2]. However, the argument given there requires some nontrivial ingredients, in particular the Arzelá-Ascoli theorem. Here we follow the approach from this lecture note and thus, paraphrasing Joe Diestel's words[3], *we should blame Manjunath Krishnapur for its simplicity.*

[2]Diestel, Joe; Spalsbury, Angela. *The joys of Haar measure.* Graduate Studies in Mathematics, 150. American Mathematical Society, Providence, RI, 2014

[3]page 7 in Diestel, Joseph, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, 92. Springer-Verlag, New York, 1984

Consider a compact group $(G,\cdot)$ with the neutral element $e\in G$.Given any open $V\ni e$, we write

$$\mathcal{H}_V=\{xVy: x,y\in G\},$$

for the family of all its translates. We say that $a,b\in G$ are V-adjacent if they belong to the same element of $\mathcal{H}_V$. A set $B\subseteq G$ is blocking if it meets every set from $\mathcal{H}_V$.

Lemma 22.1. *For every open V the family $\mathcal{H}_V$ admits a finite blocking set.*

Proof. Find open $U\ni e$ such that $U\cdot U\subseteq V$, Next pick finite $B_1,B_2\subseteq G$ such that $G=\bigcup_{x\in B_1}xU^{-1}$ and $G=\bigcup_{y\in B_2}U^{-1}y$. Then check that the set $B_1\cdot B_2$ is blocking. □

Given $f\in C(G)$, we set

$$\omega_f(V)=\sup\{|f(x)-f(y)|: x,y \text{ are } V\text{-adjacent}\}.$$

The next general fact follows from the uniform continuity of $f\in C(G)$:

Lemma 22.2. *For every $f\in C(G)$ and $\varepsilon>0$ there is open $V\ni e$ such that $\omega_f(V)\le\varepsilon$.*

For any finite nonempty set (sometimes multiset) $B\subseteq G$ we set

$$\nu_B=\frac{1}{|B|}\sum_{b\in B}\delta_b\in P(G).$$

Below we write $\nu_B(f)$ rather than $\int_G f\,\mathrm{d}\nu_B$. The following is crucial:

Theorem 22.3. *Given $V\ni e$ and two blocking sets A,B for $\mathcal{H}_V$ of the same minimal cardinality, we have*

$$|\nu_A(f)-\nu_B(f)|\le\omega_f(V),$$

for every $f\in C(G)$.

Proof. Let $A=\{a_1,\ldots,a_n\}$ and $B=\{b_1,\ldots,b_n\}$. First note that if there is a bijection $\varphi: A\to B$ such that a_i and $\varphi(a_i)$ are V-adjacent then we get the desired inequality.

To prove that such φ exists consider a bipartite graph (A,B,E) where the edge $\{a,b\}\in E$ exists iff a,b are V-adjacent. Then use the classical fact from finite combinatorics: *In a bipartite graph the maximal matching has the same size as the minimal cardinality of a blocking set.* This is the König-Egeváry theorem, see e.g. here or this text in Polish □

Theorem 22.4. *Suppose that we are given open sets $V\supseteq W\ni e$, and two minimal sets A,B that are blocking for $\mathcal{H}_W$ and $\mathcal{H}_V$, respectively. Then*

$$|\nu_A(f)-\nu_B(f)|\le 2\omega_f(V)$$

for every $f\in C(G)$.

Proof. Consider $C=A\cdot B$ (as a multiset). Then, for instance,

$$\nu_C(f)=\frac{1}{|A||B|}\sum_{a\in A,b\in B}f(ab)=\frac{1}{|A|}\sum_{a\in A}\nu_{aB}(f).$$

Note that Ab blocks $\mathcal{H}_W$ for every $b \in B$ and, conversely, aB blocks $\mathcal{H}_V$ for every $a \in A$. By Theorem 22.3,

$$|\nu_A(f) - \nu_C(f)| \le \frac{1}{|B|} \sum_{b \in B} |\nu_A(f) - \nu_{Ab}(f)| \le \omega_f(V),$$

$$|\nu_B(f) - \nu_C(f)| \le \frac{1}{|A|} \sum_{a \in A} |\nu_B(f) - \nu_{aB}(f)| \le \omega_f(V),$$

and hence $|\nu_A(f) - \nu_B(f)| \le 2\omega_f(V)$. □

Now for every open $V \ni e$ we pick a minimal blocking set $B(V)$ for $\mathcal{H}_V$ and simply write

$$\nu_V = \nu_{B(V)}.$$

Then $(\nu_V)_{V \ni e}$ is a net of measures (indexed by all open neighbourhoods of e and ordered by inverse inclusion). We conclude that such a net converges and its limit is a probability measure ν which is left- and right- invariant at the same time.

23. Uniqueness of the Haar measure

Recall here a popular argument that left invariance of a measure on a compact proup automatically implies its right-invariance; actually, the same argument gives uniqueness.

Theorem 23.1. *If ν is a left-invariant regular Borel measure on a compact group G then $\nu(U^{-1}) = \nu(U)$ for every open set $U \subseteq G$.*

Consequently, ν is also right-invariant and $\nu' = \nu$ for every left-invariant regular probability measure ν'

Proof. Consider any open $U \subseteq G$ and

$$\Gamma = \{(x, y) \in G \times G : y^{-1}x \in U\}.$$

Recall that the product measure $\nu \otimes \nu$ (defined on $Bor(G) \otimes Bor(G)$) extends to a Borel measure μ on $Bor(G \times G)$ so that μ satisifes the usual Fubini formula

$$\mu(\Gamma) = \int_G \nu(G_x)\, \mathrm{d}\nu(x) = \int_G \nu(G^y)\, \mathrm{d}\nu(y),$$

see P11/L9 (note that checking the formula for open $\Gamma \subseteq G \times G$ is simpler).

Every horizontal section Γ^y is equal to yU so it satisfies $\nu(\Gamma^y) = \nu(U)$ by left-invariance. For a vertical section Γ_x we have

$$y \in \Gamma_x \iff y^{-1} \in Ux^{-1} \iff y \in xU^{-1},$$

so $\nu(\Gamma_x) = \nu(U^{-1})$ and Fubini says $\nu(U) = \nu(U^{-1})$.

By outer regularity we have $\nu(B^{-1}) = \nu(B)$ for every Borel $B \subseteq G$. Note that, formally speaking, $B \mapsto \nu(B^{-1})$ defines a right-invariant measure which is equal to ν.

For uniqueness, apply the above Fubini-like argument to $\nu \otimes \nu'$. □

Remark 23.2. (1) There is a nontrivial theorem stating that every Haar measure satisfies the assertion of Theorem 18.2, that $Bor(G)$ is contained in the completion of $Baire(G)$ with respect to the Haar measure.

(2) If G is infinite discrete group then the existence of an invariant finitely additive probability on (the power set of) G is another interesting story. A group is said to be *amenable* if it admits such a function. We have already seen (L7/P2) that $(\mathbb{Z}, +)$ is amenable (actually, every abelian group is amenable).

(3) The classical non-amenable group is F_2, the free group of two generators. If you cannot find a reason why F_2 does not admit an invariant probability (and you cannot fall asleep for that reason) then check it in WIKIPEDIA or look into this introductory text by Alejandra Garrido