Generalisations of ϵ-density

Mirna Džamonja
School of Mathematics
University of East Anglia
Norwich, NR4 7TJ, UK

h020@uea.ac.uk

Grzegorz Plebanek
Institute of Mathematics
University of Wrocław
2/4 Pl. Grunwaldzki
50384 Wrocław, Poland

grzes@math.uni.wroc.pl

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Abstract

We give several partial solutions to Fremlin's question DU about the existence of large homogeneous sets for ε–dense open families of finite sets of ordinals by introducing and considering some generalisations of the notion of ε–density. In particular we prove that every $\frac{1}{2}$–functionally dense open family has an infinite homogeneous set and that under $MA+\neg CH$ every $\frac{1}{2}$–dense open family satisfying an additional covering property is $\frac{1}{2}$–functionally dense and has a homogeneous set of size $\aleph_1$. Moreover, we prove that assuming $MA+\neg CH$ satisfying this covering property on a set of size $\aleph_1$ is in fact a necessary condition for a $\frac{1}{2}$–dense open family to have a homogeneous set of size $\aleph_1$.

We also study the continuos version of ε–density and give some negative homogeneity results.

0 Introduction

Several years ago D.H. Fremlin formulated a question motivated by measure theory but quite combinatorial in nature, that has to do with the existence of large homogeneous

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sets for certain families of finite sets. The question, denoted DU on his list of problems, see [4], is formulated as follows.

Definition 0.1 *Let A be any infinite set and let $\varepsilon \in (0,1)$.*
(1) A family $\mathcal{D} \subseteq [A]^{<\aleph_0}$ is said to be ε–dense on A *if for every $B \in [A]^{<\aleph_0}$ there is $D \in \mathcal{D}$ such that $D \subseteq B$ and $|D| \geq \varepsilon |B|$.*
(2) A family $\mathcal{D} \subseteq [A]^{<\aleph_0}$ is said to be open *if it is closed under taking subsets.*
(3) If $\mathcal{D} \subseteq [A]^{<\aleph_0}$ we say that $A^* \subseteq A$ is homogeneous for $\mathcal{D}$ *if $[A^*]^{<\aleph_0} \subseteq \mathcal{D}$.*

Problem 0.2 (DU) *(i) If $\mathcal{D}$ is an ε–dense open on ω_1, is there necessarily an infinite subset of ω_1 which is homogeneous for $\mathcal{D}$?*
(ii) Assume $MA(\aleph_1) + \neg CH$ and that $\mathcal{D}$ is ε–dense open on ω_1. Is there necessarily a homogeneous set for $\mathcal{D}$ of size $\aleph_1$?

It is well known that there is a $\frac{1}{2}$–dense open family on ω without an infinite homogeneous set (Schreier family) and that under CH there is a $\frac{1}{2}$–dense open family on ω_1 without a homogeneous set of size $\aleph_1$. It was shown by Fremlin in [4] that in fact the answer to Problem DU will remain the same if one replaces ε throughout by $1/2$. We shall hence concentrate on $\frac{1}{2}$–dense open families.

Although the question has been around for quite a while and many attempts have been made to solve it (e.g. [1], [4]) only partial solutions are available. It for one appears that the kind of density demanded in this question is rather intractable by the methods one would use in combinatorial set theory to handle some more familiar density notions. Given this somewhat notorious status of the question it seems that even a partial solution offering another way of looking at the problem may be of some interest, and it is with this in mind that this note is written. In it we deal with two generalisations of ε-density which both give the original notion as a special case. The first one is the notion of functional density, and it is described in §1. As shown in there the existence of infinite homogeneous sets for this kind of dense families can be proved outright in ZFC, and moreover the notion of measure precalibres gives rise to larger homogeneous sets. Hence this gives a partial affirmative answer to DU(i) under the additional assumption of functional density. In §2 we use functional density to give a partial answer to DU (ii), in the affirmative under an additional assumption formulated not as a density condition but as a covering property. This property says that in some rather weak sense the family is close to being closed under finite unions (in the latter case of course the homogeneous set is easily seen to exist). Our approach allows us to state a characterisation under $MA(\aleph_1)$ of those $\frac{1}{2}$–dense open families on ω_1 that admit a homogeneous set of size $\aleph_1$, phrased in terms of the covering property used in the main proof.

Section 3 is of quite a different flavour. In it we notice that ε–density may be understood as a specialisation to the counting measure of a general measure-theoretic density notion. We show some examples of families that are dense in this more general sense and do not admit homogenous sets.

1 Functionally dense families

Definition 1.1 *Let A be any infinite set. A family $\mathcal{D} \subseteq [A]^{<\aleph_0}$ is said to be* $\frac{1}{2}$–functionally dense *or* $\frac{1}{2}$–FD *on A if for every $B \in [A]^{<\aleph_0}$ and every function $g : B \to \omega$ there is $D \in \mathcal{D}$ such that $D \subseteq B$ and $\sum_{\alpha\in D} g(\alpha) \geq \frac{1}{2}\sum_{\alpha\in B} g(\alpha)$.*

Observation 1.2 *Any family that is $\frac{1}{2}$–FD on A is $\frac{1}{2}$–dense on A, as can be seen by considering the constant 1 function. The converse is not true, since the family of all finite $D \subseteq (\omega \setminus \{0\})$ that satisfy $\min(D) \geq |D|$ is known to be $\frac{1}{2}$–dense on $\omega \setminus \{0\})$ but the function g assigning 1 to 1 and 2 and 0 to 3 shows that the family is not $\frac{1}{2}$–functionally dense on $\omega \setminus \{0\}$.*

The following theorem giving a positive homogeneity result shows that the notion of functional density corresponds to the usual Kelley's intersection number ([2] in the sense that epsilon density corresponds to the weak Kelley intersection number (see [4] and [5] for the latter notion and its connection with epsilon density). Although we shall assume familiarity with the well known notions of Kelley's intersection number and Kelley's theorem, the notion of a measure precalibre might be less familiar so we remind the reader of what its definition. An infinite cardinal κ is called *measure precalibre* iff for every family of κ sets in the measure algebra of 2^κ whose measure is uniformly bounded away from 0, there is a centred subfamily of size κ. One can consult [3] for further equivalent definitions and also to note that every cardinal that is a precalibre of measure algebras is a measure precalibre.

Theorem 1.3 *Suppose that $\mathcal{D} \subseteq [\kappa]^{<\aleph_0}$ is $\frac{1}{2}$–FD on κ and open. Then there is an infinite set $X \subseteq \kappa$ which is $\mathcal{D}$–homogenous. If κ is a measure precalibre then there is a $\mathcal{D}$–homogenous set of size κ.*

Proof. By identifying sets in $\mathcal{D}$ with their characteristic functions and ${}^\kappa 2$ with 2^κ we may treat $\mathcal{D}$ as a subset of 2^κ. Let $A_\alpha := \{D \in \mathcal{D} : \alpha \in d\}$ for $\alpha < \kappa$. By a straightforward argument we check that the family $(A_\alpha)_{\alpha<\kappa}$ has Kelley's intersection number at least $\frac{1}{2}$. The proof of Kelley's theorem [2] gives a finitely additive measure μ on 2^κ with $\mu(A_\alpha) \geq \frac{1}{2}$ for every α. Since $\aleph_0$ is a measure precalibre there is an infinite set $X \subseteq \kappa$ for which $(A_\alpha)_{\alpha\in X}$ has the finite intersection property. This exactly means that $[X]^{<\omega} \subseteq D$.

The last statement of the theorem follows similarly. $\bigstar_{1.3}$

It turns out that for any κ and a $\frac{1}{2}$–dense family $\mathcal{D} \subseteq [\kappa]^{<\aleph_0}$ one may define a $\frac{1}{2}$–FD family $\mathcal{C}(\mathcal{D})$ on κ that is highly connected with the original family.

Lemma 1.4 *Suppose that $\mathcal{D} \subseteq [\kappa]^{<\aleph_0}$ is $\frac{1}{2}$–dense on κ and that $h : \kappa \to \kappa \times \omega$ is a bijection. Let D be the family of all images under h of the elements of $\mathcal{D}$. Then*

$$\mathcal{C}(\mathcal{D}) := \{c \in [\kappa]^{<\aleph_0} : (\exists d \in D)c = \pi(d)\}$$

is $\frac{1}{2}$–FD on κ, where $\pi : \kappa \times \omega \to \kappa$ is the standard projection on the first coordinate.

Proof. Note that the definition of 1/2-FD on κ does not change if we consider only functions $g : B \to \omega \setminus \{0\}$ for all $B \in [\kappa]^{<\aleph_0}$, and also note that D is $\frac{1}{2}$–dense on $\kappa \times \omega$. So let $B \in [\kappa]^{<\aleph_0}$ and $g : B \to \omega \setminus \{0\}$ be given. Consider the set

$$A = \bigcup_{\alpha \in B} \{\alpha\} \times \{1, 2, \dots, g(\alpha)\}.$$

Since D is $\frac{1}{2}$–dense on $\kappa \times \omega$, we may find $d \in D$ such that $d \subseteq A$ and $|d| \geq \frac{1}{2}|A|$. Then, putting $c = \pi(d)$, we get $c \subseteq B$, and

$$\sum_{\alpha \in c} g(\alpha) \geq |d| \geq \frac{1}{2}|A| = \frac{1}{2} \sum_{\alpha \in B} g(\alpha).$$

$\bigstar_{1.4}$

It is not clear if one can use large homogeneous sets for $\mathcal{C}(\mathcal{D})$ to construct homogeneous sets for $\mathcal{D}$, but at least when $\aleph_1$ is a precalibre of measure algebras (in particular when MA($\aleph_1$) holds) the discussion above leads to a reformulation of the DU problem that at least appears to be simpler than the original.

Theorem 1.5 *Assume that $\aleph_1$ is a precalibre of measure algebras and that every family D that is $\frac{1}{2}$–dense open in $\omega_1 \times \omega$, and satisfies $\{\pi(d) : d \in D\} = [\omega_1]^{<\aleph_0}$ admits a homogeneous set of size λ. Then every $\frac{1}{2}$–dense open family on ω_1 admits a homogeneous set of size λ.*

Proof. Let $\mathcal{D}$ be a $\frac{1}{2}$–dense family on ω_1 and let h be any bijection between ω_1 and $\omega_1 \times \omega$. We construct $\mathcal{C}(\mathcal{D})$ as in Lemma 1.4, so a $\frac{1}{2}$–FD family on ω_1. By the same lemma, Theorem 1.3 and the assumption that $\aleph_1$ is a precalibre of measure algebras yields a set $Z \in [\omega_1]^{\aleph_1}$ such that Z is homogeneous for $\mathcal{C}(\mathcal{D})$. Using a bijection between Z and ω_1 we can assume that $Z = \omega_1$. Then D as in the proof of Lemma 1.4 satisfies the assumptions of this lemma and any set that is homogeneous for it gives a homogeneous set for $\mathcal{D}$ of the same cardinality as the original. $\bigstar_{1.5}$

In the next section we show that under MA($\aleph_1$) every $\frac{1}{2}$–dense open family on ω_1 satisfying a certain covering property is actually $\frac{1}{2}$–FD , and hence admits an uncountable homogeneous subset.

2 A covering property

When trying to construct homogeneous sets for dense open families the major difficulty is that except in trivial cases the family is not closed under finite unions. This means that one cannot assume too much in the way of a covering property of such a family. One may wonder if some weak covering property would suffice and in this section we consider a nontrivial such property. The property considered here is such that any $\frac{1}{2}$–dense open

family on ω_1 satisfying it can be by a ccc forcing made into an $\frac{1}{2}$–FD open family on ω_1. This is interesting if one studies the behaviour of $\frac{1}{2}$–dense open families on ω_1 under $MA(\aleph_1)$.

Definition 2.1 *For a Δ-system $\bar{F} = \langle F_\alpha : \alpha < \omega_1 \rangle$ of finite sets of ordinals with root r we say that $\bar{F}$ is* clean *iff*

(1) $\max(r) < \min(F_\alpha \setminus r)$ *for all* α,

(2) $\max(F_\alpha) < \min(F_\beta \setminus r)$ *for* $\alpha < \beta < \omega_1$,

(3) $|F_\alpha|$ *is a constant.*

Definition 2.2 *Suppose that $\mathcal{D}$ is a $\frac{1}{2}$–dense open family of finite subsets of ω_1. We say that $\mathcal{D}$ satisfies the* $\mathrm{cv}(4,2)$-condition *iff for every two clean Δ-systems $\bar{F}^l$ $(l < 2)$ of elements of $\mathcal{D}$, there is $\alpha \in [1, \omega_1)$ and $H_0, H_1 \in \mathcal{D}$ such that*

$$(F^0_\emptyset \cup F^0_\alpha) \cup (F^1_0 \cup F^1_\alpha) \subseteq H_0 \cup H_1.$$

Theorem 2.3 *Assume that $MA(\aleph_1)$ holds. Then for every $\frac{1}{2}$–dense open family $\mathcal{D}$ of finite subsets of ω_1 that satisfies the* $\mathrm{cv}(4,2)$*-condition, there is $A^* \in [\omega_1]^{\aleph_1}$ such that $[A^*]^{<\aleph_0} \subseteq \mathcal{D}$.*

Proof. Let $\mathcal{D}$ be as in the assumptions. We define a forcing notion $\mathbb{P}$ by letting its universe be the set $\{F_0 \cup F_1 : F_0, F_1 \in \mathcal{D}\}$ and ordering it by $p \leq q$ (q is stronger) iff $p \subseteq q$.

Claim 2.4 *For each $\beta < \omega_1$ the set*

$$\mathcal{E}_\beta \stackrel{\mathrm{def}}{=} \{p \in \mathbb{P} : \max(p) \geq \beta\}$$

is dense in $\mathbb{P}$.

Claim 2.5 *Let $p \in \mathbb{P}$ be given, and let $\beta < \omega_1$. We commence by finding $F_0, F_1 \in \mathcal{D}$ such that $p = F_0 \cup F_1$. Let $n_l = |F_l|$ for $l < 2$. For each $l < 2$ we construct by induction on $\alpha < \omega_1$ a sequence $\bar{F}^\alpha = \langle F^l_\alpha : \alpha < \omega_1 \rangle$ of elements of $\mathcal{D}$ such that*

(1) F^l_0 *is given,*

(2) If $\alpha < \alpha'$, *then* $\max(F^l_\alpha) < \min(F^l_{\alpha'})$,

(3) If $\alpha > 0$ *then* $\max(F^l_\alpha) \geq \beta$,

(4) $|F^l_\alpha| = n_l$.

Such sequences can be easily defined by using the 1/2-density and openness of $\mathcal{D}$. Using the property $\mathrm{cv}(4,2)$, *there is $\alpha \geq 1$ and $H_0, H_1 \in \mathcal{D}$ such that $p \cup F^0_\alpha \cup F^1_\alpha \subseteq H_0 \cup H_1$. Letting $q = H_0 \cup H_1$ we obtain a condition in $\mathcal{E}_\beta$ that extends p.* $\bigstar_{2.4}$

Claim 2.6 $\mathbb{P}$ *is ccc.*

Proof of the Claim. Suppose we are given $\{p_\alpha : \alpha < \omega_1\}$ from $\mathbb{P}$ and let us choose for each α a pair (F_0^α, F_1^α) of elements of $\mathcal{D}$ such that $p_\alpha = F_0^\alpha \cup F_1^\alpha$. By applying the Δ-system lemma twice, we may assume that $\bar{F}^l = \langle F_\alpha^l : \alpha < \omega_1 \rangle$ forms a clean Δ-system, for $l < 2$. By the $\mathrm{cv}(4,2)$ condition there is $\alpha < \omega_1$ and $H_0, H_1 \in \mathcal{D}$ such that $(F_0^0 \cup F_\alpha^0) \cup (F_0^1 \cup F_\alpha^1) \subseteq H_0 \cup H_1$, so p_0 and p_α are compatible in $\mathbb{P}$. $\bigstar_{2.6}$

By $MA(\aleph_1)$ we can find a filter G in $\mathbb{P}$ that intersects all $\mathcal{E}_\beta$. Let $A \stackrel{\mathrm{def}}{=} \bigcup G$.

Claim 2.7 $A \in [\omega_1]^{\aleph_1}$ *and it satisfies that for every* $F \in [A]^{<\aleph_0}$*, there are* $F_0, F_1 \in \mathcal{D}$ *such that* $F = F_0 \cup F_1$.

Proof of the Claim. If $p \in G \cap \mathcal{E}_\beta$, then $\max(p) \geq \beta$, so by the genericity of G we obtain that $\sup(A) = \omega_1$.

Suppose that $F \in [A]^{<\aleph_0}$, hence there are $q_0, \ldots q_n \in G$ with $F \subseteq \cup_{i \leq n} q_i$. As G is a filter, conditions $q_0, \ldots q_n$ have a common upper bound in G, so without loss of generality $n = 0$. Let $q_0 = H_0 \cup H_1$ for some $H_0, H_1 \in \mathcal{D}$, and let $F_l = H_l \cap F$. $\bigstar_{2.7}$

Now notice that $\mathcal{D}$ is $\frac{1}{2}$–FD on A, since if $F \in [A]^{<\aleph_0}$ and $g : F \to \omega$, choosing $F_0, F_1 \in \mathcal{D}$ so that $F = F_0 \cup F_1$, we obtain

$$\Sigma_{\alpha \in F} g(\alpha) \leq \Sigma_{\alpha \in F_0} g(\alpha) + \Sigma_{\alpha \in F_1} g(\alpha),$$

so for at least one $l \in 2$ we have $\Sigma_{\alpha \in F_l} g(\alpha) \geq 1/2 \, \Sigma_{\alpha \in F} g(\alpha)$. Hence by Theorem 1.3 and the fact that $MA(\aleph_1)$ implies that $\aleph_1$ is a precalibre of measure algebras, we obtain that there is $A^* \in [A]^{\aleph_1}$ which is $\mathcal{D}$-homogeneous. $\bigstar_{2.3}$

It might be interesting to note that the full force of the covering condition used in Theorem 2.3 was not needed for the chain condition, but it was for the density argument. We may now use Theorem 2.3 and its proof to obtain a characterisation under $MA(\aleph_1)$ of those $\frac{1}{2}$–dense open families on ω_1 that have a homogeneous set of size $\aleph_1$.

Corollary 2.8 *Assume* $MA(\aleph_1)$*. Then a* $\frac{1}{2}$*–dense open family* $\mathcal{D}$ *on* ω_1 *has a homogeneous set of size* $\aleph_1$ *iff for some set* $X \in [\omega_1]^{\aleph_1}$ *the family* $\mathcal{D} \cap [X]^{<\aleph_0}$ *satisfies the* $\mathrm{cv}(4,2)$ *condition.*

Proof. In the forward direction, let X be the homogeneous set for $\mathcal{D}$ that has size $\aleph_1$. In the backward direction, using the same forcing as in the proof of Theorem 2.3 with $\mathcal{D} \cap [X]^{<\aleph_0}$ in place of $\mathcal{D}$ we may notice that the only relevant instances of the covering property are those applicable to $\mathcal{D} \cap [X]^{<\aleph_0}$, and hence the forcing argument will produce a $Y \in [X]^{\aleph_1}$ which is homogeneous for the family $\mathcal{D}$ restricted to X and hence for $\mathcal{D}$. $\bigstar_{2.8}$

3 A continuous version of the problem

One way to understand problem DU is to view it in the context of the counting measure, as we shall explain below. This approach suggests a generalisation to a more general measure-theoretic context and we attempt such a thing here. For the rest of the section we work in the context of a given (X, Σ, ν) which an infinite but semi–finite measure space. Hence we assume that $\nu(X) = \infty$ but for every $E \in \Sigma$ we have

$$\nu(E) = \sup\{\nu(A) : \ A \in \Sigma, \nu(A) < \infty\}.$$

Definition 3.1 *We say that $\mathcal{D} \subseteq \Sigma$ is $\frac{1}{2}$–dense open in Σ if it satisfies the following*

(i) if $C \subseteq D \in \mathcal{D}$ and $C \in \Sigma$ then $C \in \mathcal{D}$;

(ii) if $E \in \Sigma$ and $\nu(E) < \infty$ then there is $D \in \mathcal{D}$ such that $D \subseteq E$ and $\nu(D) \geq \frac{1}{2}\nu(E)$.

In this context we ask the following

Problem 3.2 (generalised DU) *Does every ε–dense family $\mathcal{D} \subseteq \Sigma$ contain a $\subseteq$-increasing sequence (D_n) with $\lim \nu(D_n) = \infty$?*

Taking $X = \kappa$, $\Sigma = \mathcal{P}(\kappa)$ and ν the counting measure on κ we obtain the original problem DU (about infinite homogenous subsets) from the generalised DU. Since in this context we can only discuss homogeneous sets of size $\aleph_0$ the possible interest of the generalised DU lies in purely ZFC results. It turns out that counterexamples are easier to produce in this setting. It is to be expected that the idea of Schreier family on ω which gives a counterexample to the original DU setting may be adopted to another example, and that is what we shall do here.

Example 3.3 *Let λ to be the Lebesgue measure on $X = [0, \infty)$ and let $\mathcal{D}$ be a family of those Borel sets D, such that either $D = \emptyset$ or $\lambda(D) \leq \inf(D)$. Clearly $\mathcal{D}$ is open. For any Borel set $E \subseteq X$ of finite measure there is a point x_0 such that $\lambda([0, x_0] \cap E) \geq \frac{1}{2}\lambda(E)$. Writing $D = [x_0, \infty] \cap E$ we have $D \in \mathcal{D}$, so $\mathcal{D}$ is $\frac{1}{2}$–dense. Of course, there is no $\subseteq$-increasing sequence $\langle D_n : n < \omega\rangle$ with $\lambda(D_n) \to \infty$.*

The above example of course gives a measure of a countable Maharam type. The following example has Maharam type $\mathfrak{c}$.

Example 3.4 *Let $X = \mathbb{R} \times \{0,1\}^{\mathfrak{c}}$ and let ν be the product of the Lebesgue measure λ and the usual product measure μ, where μ is defined on the product σ–algebra of $\{0,1\}^{\mathfrak{c}}$. We let Σ be the domain of ν; then Σ is of size $\mathfrak{c}$ and every $E \in \Sigma$ is determined by $\mathbb{R}$ and countably many coordinates in $\{0,1\}^{\mathfrak{c}}$. Let $\{E_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all sets in Σ of finite measure, modulo measure 0, and for each $\xi < \mathfrak{c}$ let I_ξ be a countable subset of $\mathfrak{c}$ such that E_ξ is determined by $\mathbb{R}$ and the coordinates in I_ξ.*

By induction on $\xi < \mathfrak{c}$ we may define a function $\theta : \mathfrak{c} \to \mathfrak{c}$ such that

$$\theta(\xi) \notin \bigcup_{\alpha \leq \xi} I_\xi \cup \{\theta(\alpha) : \alpha < \xi\}$$

for every $\xi < \mathfrak{c}$. Considering elements of X as sequences with $1 + \mathfrak{c}$ entries and calling the first entry -1, we may define for $\xi < \mathfrak{c}$ sets $C_\xi = \{x \in X : x(\xi) = 0\}$. It follows from the definition of μ and ν that $\nu(\bigcap_{k \geq 1} C_{\xi_k}) = 0$ whenever $\langle \xi_k : k < \omega \rangle$ is a sequence of distinct elements of $\mathfrak{c}$. $(*)$

Now let $D_\xi = E_\xi \cap C_{\theta(\xi)}$ for every $\xi < \mathfrak{c}$, and define $\mathcal{D}$ to be a family of those $D \in \Sigma$ which are contained in some D_ξ. It follows that $\mathcal{D}$ is $\frac{1}{2}$–dense open, since

$$\nu(D_\xi) = \nu(E_\xi \cap C_{\theta(\xi)}) = \frac{1}{2}\nu(E_\xi).$$

Note that the last conclusion uses the fact that $\theta(\xi) \notin I_\xi$. Now we claim that there is no increasing sequence $\langle A_n : n < \omega \rangle$ in $\mathcal{D}$ with $\lim \nu(A_n) = \infty$.

So suppose otherwise. Take n_1 such that $\nu(A_{n_1}) \geq 1$. Since $A_{n_1} \in \mathcal{D}$ then $A_{n_1} \subseteq D_{\xi_1}$ for some ξ_1. But $\nu(D_{\xi_1}) < \infty$ so there is $n_2 > n_1$ such that $\nu(A_{n_2}) > \nu(D_{\xi_1})$. Now $A_{n_2} \subseteq D_{\xi_2}$ for some ξ_2, where $\xi_2 \neq \xi_1$.

In this way we can define $n_1 < n_2 < \ldots$, and distinct $\xi_k < \mathfrak{c}$ such that $A_{n_k} \subseteq D_{\xi_k} \subseteq C_{\theta(\xi_k)}$. Hence $A_{n_1} \subseteq P$, where $P = \bigcap_k C_{\theta(\xi_k)}$. But since θ is 1-1 it follows by $(*)$ above that $\nu(P) = 0$, which is a contradiction.

The measure from Example 3.4 is of type $\mathfrak{c}$ and σ-finite. One might however modify the construction to get a non–σ–finite example (replacing λ on the first axis by any non–σ–finite measure on some σ–algebra of size $2^{\mathfrak{c}}$ and letting μ be the product measure on $\{0,1\}^{2^{\mathfrak{c}}}$). The examples above leave open the generalised DU question for those ν that are of countable type on every set of finite measure but not σ–finite.

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