On isomorphic embeddings of $C(K)$ spaces

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General question

Suppose that K and L are compact spaces and there is an isomorphic embedding $C(K) \hookrightarrow C(L)$. How K is related to L?

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A concrete problem

Suppose that L is Corson compact and $C(K) \hookrightarrow C(L)$. Must K be Corson compact?

Terminology

A compact space K is Corson compact (denoted $K \in \mathfrak{C}$) if K can be topologically embedded, fo some κ, into

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Basic facts

- **Amir & Lindenstrauss:** If L is Eberlein compact then every K such that $C(K) \hookrightarrow C(L)$ is Eberlein compact.
- Every Eberlein compact embeds into $c_0(\kappa) \subseteq \Sigma(\mathbb{R}^\kappa)$ so is Corson compact.
- **Argyros et al.:** Assuming MA + nonCH, if $L \in \mathfrak{C}$ then $M_1(K) \in \mathfrak{C}$. Consequently, if $L \in \mathfrak{C}$ and $C(K) \hookrightarrow C(L)$ then $K \in \mathfrak{C}$, too.

Notation

$\mathfrak{C}$ is the class of Corson compacta; denote by $\mathfrak{C}^*$ those compact spaces K for which there is $L \in \mathfrak{C}$ such that $C(K) \hookrightarrow C(L)$.

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- Is K='the double arrow space' in $\mathfrak{C}^*$?

Countably determined measures

A measure $\mu \in P(K)$ (i.e. a probability regular Borel measure on K) is **countably determined** if there is a countable family $\mathcal{F}$ of closed subsets of K such that

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Some partial answers (Marciszewski & GP)

Theorem. *Suppose that every $\mu \in P(K)$ is countably determined. Then $K \in \mathfrak{C}^*$ implies $K \in \mathfrak{C}$.*

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- *K is linearly ordered, or*
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- *K is Rosenthal compact.*

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Classical results

- **Banach-Stone:** *If $C(K)$ and $C(L)$ are isometric then $K \cong L$.*
- **Kaplansky:** *If $C(K)$ and $C(L)$ are isomorphic as Banach lattices then $K \cong L$.*
- **Amir, Cambern:** *If $T : C(K) \to C(L)$ is an isomorphism onto and $||T|| \cdot ||T^{-1}|| < 2$ then $K \cong L$.*

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Theorem 2. *Suppose that* K *is zerodimensional and there is a positive embedding* $T : C(K) \to C(L)$. *Then there is a* π*-base of* K, *consisting of clopen subsets which are continuous images of some closed subspaces of* L.

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What about 2^{ω_1}?

There is no positive embedding $T : C(2^{\omega_1}) \to C(L)$ with $L \in \mathfrak{C}$.

Can one show in ZFC that $2^{\omega_1} \notin \mathfrak{C}^*$?