On compact spaces carrying Radon measures of large Maharam type

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Abstract

Assuming that a compact space K carries a Radon measure of an uncountable Maharam type κ, we investigate if the space $P(K)$ (of probability measures on K) or the space K itself can be continuosly mapped onto the cube $[0,1]^\kappa$, and consider related results on the isomorphic structure of a Banach space $C(K)$.

We present here a short survey of this subject, in some cases presenting new proofs or improvements of known results. This new approach is based on a measure theoretic lemma due to Fremlin and Haydon.

1 Introduction.

For a compact space K we denote by $C(K)$ the Banach space of continuous functions on K and by $P(K)$ the space of all probability Radon measures defined on K. We always consider $P(K)$ equipped with the *weak** topology inherited from $C(K)^*$.
Given infinite cardinal numbers κ and a compact space K, we consider the following statements on K, $C(K)$ and $P(K)$.

A(κ, K) *there is a continuous surjection from K onto $[0,1]^\kappa$;*

B(κ, K) *$l^1(\kappa)$ can be isomorphically embedded into $C(K)$;*

C(κ, K) *K carries a homogenous Radon measure of type κ;*

D(κ, K) *there is a continuous surjection from $P(K)$ onto $[0,1]^\kappa$.*

It is our aim here to discuss which implications between the above statements are true for all compact spaces K (when κ is fixed). Therefore for any X and Y from $\{$A, B, C, D$\}$, we use the convention that

$$\mathrm{X}(\kappa) \Rightarrow \mathrm{Y}(\kappa) \quad \text{means} \quad \mathrm{X}(\kappa, K) \Rightarrow \mathrm{Y}(\kappa, K) \text{ for every compact } K.$$

2000 *Mathematics Subject Classification*: Primary 28C15, 54A25; Secondary 46B46.

Partially supported by KBN grant 5 P03A 037 20. The research done while the author held a visiting post at the Institute of Mathematics, Polish Academy of Sciences.

Thanks to numerous results on $\mathrm{A}(\kappa), \mathrm{B}(\kappa), \mathrm{C}(\kappa)$ and $\mathrm{D}(\kappa)$ that have been published by several mathematicians we now have an almost complete picture of their mutual relationships. Section 2 of the present paper contains a short survey of the subject. The account of the (mostly standard) terminology and notation used here is given in section 3, where we also quote some auxiliary results needed in the sequel.
With the exception of section 6 we present no new results here. We show, however, that the proofs of the implications $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$, $\mathrm{C}(\kappa) \Rightarrow \mathrm{B}(\kappa)$, $\mathrm{C}(\kappa) \Rightarrow \mathrm{D}(\kappa)$, where $\kappa \geq \omega_2$ may be derived from a single measure–theoretic lemma due to Fremlin and Haydon which is presented in section 4. This enables one to considerably simplify previously known arguments (see section 5). In section 6 we use the notion of an m–precaliber of measure algebras introduced by Haydon to extend the result on the equivalence $\mathrm{A}(\kappa) \Leftrightarrow \mathrm{C}(\kappa)$ to the case of an aritrary cardinal number $\kappa \geq \omega_2$.

Acknowledgements. The content of this paper was presented during the 30th Winter School in Abstract Analysis (Lhota, January 2002). I wish to thank the organizers of the conference for their hospitality and for the financial support of my stay in Czech Republic.
I am very grateful to Richard Haydon for his permission to reproduce the proof of Lemma 4.1 here, and to Mirna Džamonja for very useful remarks concerning the subject.

2 A survey of A, B, C, D.

We shall first point out the implications that are true for every κ:

$$\mathrm{A}(\kappa) \Rightarrow \mathrm{B}(\kappa) \Rightarrow \mathrm{C}(\kappa), \quad \mathrm{A}(\kappa) \Rightarrow \mathrm{D}(\kappa).$$

Indeed, if K can be continuously mapped onto $[0,1]^\kappa$ then $C([0,1]^\kappa)$ is isometric to a subspace of $C(K)$, so $\mathrm{A}(\kappa) \Rightarrow \mathrm{B}(\kappa)$ follows from the fact that $C([0,1]^\kappa)$ contains an isometric copy of $l^1(\kappa)$.
$\mathrm{B}(\kappa) \Rightarrow \mathrm{C}(\kappa)$ is a particular case of a result due to Pełczyński [20], stating that for every Banach space E, if $l^1(\kappa) \hookrightarrow E$ then $L^1(\mu_\kappa) \hookrightarrow E^*$, where $\hookrightarrow$ denotes an isomorphic embedding and μ_κ is the usual product measure on $\{0,1\}^\kappa$. Indeed, $L^1(\mu_\kappa) \hookrightarrow C(K)^*$ is equivalent to saying that K carries a homogenous Radon measure of type κ, see [13], Remarks 2.5.
Recall also a direct argument for $\mathrm{A}(\kappa) \Rightarrow \mathrm{C}(\kappa)$. If $g : K \to [0,1]^\kappa$ is a continuous surjection then there is $\mu \in P(K)$ such that $g(\mu) = \mu_\kappa$ and such a measure μ can be taken so that it is homogenous of type κ, see Haydon [13], Proposition 2.1.
$\mathrm{A}(\kappa) \Rightarrow \mathrm{D}(\kappa)$ follows simply from the observation that K can be treated as a closed subspace of $P(K)$ and every continuous mapping on K can be extended to $P(K)$.

Other implications hold only under some restrictions on κ and sometimes can be proved only under additional set–theoretic axioms.
The question if $\mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa)$ is a particular case of a problem posed by Pełczyński, if for every Banach space E, $L^1(\mu_\kappa) \hookrightarrow E^*$ implies $l^1(\kappa) \hookrightarrow E$.

Pełczyński [20] showed that this is true for $\kappa = \omega$, so in particular $\mathrm{C}(\omega) \Leftrightarrow \mathrm{B}(\omega)$.
This equivalence is a consequence of a classical result that $\mathrm{C}(\omega) \Rightarrow \mathrm{A}(\omega)$. Recall that if K carries a homogenous Radon measure of type ω then K is not scattered, i.e. K can be continuously mapped onto $[0,1]$, which in turn admits a continuous surjection onto $[0,1]^\omega$, (see e.g. [25]).
Hence we have $\mathrm{A}(\omega) \Leftrightarrow \mathrm{B}(\omega) \Leftrightarrow \mathrm{C}(\omega)$. Note that $\mathrm{D}(\omega)$ is satisfied whenever $|K| > 1$, so any nontrivial scattered space K shows that $\mathrm{D}(\omega)$ does not imply $\mathrm{A}(\omega)$.
R. Haydon originated an investigation of the uncountable cardinal numbers κ for which the implication $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$ holds. Haydon [13] proved that $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$ holds for every κ with the property that $\tau < \kappa$ implies $\tau^\omega < \kappa$; in particular $\mathrm{C}(\mathfrak{c}^+) \Rightarrow \mathrm{A}(\mathfrak{c}^+)$.
Assuming the continuum hypothesis (CH) Haydon [14] proved, however, that $\mathrm{C}(\omega_1)$ implies neither $\mathrm{B}(\omega_1)$ nor $\mathrm{A}(\omega_1)$. In fact Haydon's space K form [14] has the property that $|P(K)| = \omega_1$ so it also shows that $\mathrm{C}(\omega_1) \Rightarrow \mathrm{D}(\omega_1)$ does not hold under CH.
Pełczyński's conjecture mentioned above was solved by Argyros [1]. In particular, Argyros showed that $\mathrm{C}(\kappa) \Rightarrow \mathrm{B}(\kappa)$ for every $\kappa \geq \omega_2$ and $\mathrm{C}(\omega_1) \Rightarrow \mathrm{B}(\omega_1)$ is true under MA+$\neg$CH.
Haydon's example from [14] of a compact space that cannot be mapped onto $[0,1]^{\omega_1}$ but carries a Radon measure of uncountable type initiated several constructions of "small" compact spaces with measures of uncountable type, see Talagrand [27], Kunen [17], Džamonja & Kunen [5], Plebanek [21], Kunen & van Mill [18], Plebanek [24]. Recall for instance that under the assumption that ω_1 is not a precaliber of measure algebras there is a Corson compact first–countable space admitting a Radon measure of uncountable type, see [18], cf. [24].
The author proved in [24] that, given $\kappa \geq \mathrm{cf}(\kappa) \geq \omega_2$, the implication $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$ holds if and only if κ is a precaliber of measure algebras. This means in particular, that $\mathrm{C}(\mathfrak{c}) \Rightarrow \mathrm{A}(\mathfrak{c})$ is relatively consistent with ZFC. This result was extended by Haydon [15] to the case of $\kappa \geq \omega_2$ of uncountable cofinality (see Theorem 5.1 below). R. Haydon, moreover, suggested a modification of the notion of precaliber which is applicable to κ with $\mathrm{cf}(\kappa) = \omega$ (see section 6).
Fremlin [9] solved the so–called Haydon problem showing that $\mathrm{C}(\omega_1) \Rightarrow \mathrm{A}(\omega_1)$ under MA+$\neg$CH (in fact proving that under Martin's axiom $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$ for every $\kappa < \mathfrak{c}$).
It follows from a result due to Talagrand [28] that $\mathrm{B}(\kappa) \Leftrightarrow \mathrm{D}(\kappa)$ for every κ of uncountable cofinality, see also section 3 of [19], where a shorter proof due to Argyros is presented. We prove here (Theorem 5.2) that $\mathrm{C}(\kappa) \Rightarrow \mathrm{D}(\kappa)$ whenever $\kappa \geq \omega_2$. It follows that $\mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa) \Leftrightarrow \mathrm{D}(\kappa)$ for all $\kappa \geq \omega_2$ of uncountable cofinality.
We summarize below what is known for $\kappa = \omega, \omega_1, \omega_2$, where the entry "precal" denotes the assumption that ω_2 is a precaliber of measure algebras, and it should be understood that none of the missing implications holds under the axiom given.

κ	under	implications
ω	in ZFC	$\mathrm{A}(\kappa) \Leftrightarrow \mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa) \Rightarrow \mathrm{D}(\kappa)$
ω_1	CH	$\mathrm{A}(\kappa) \Rightarrow \mathrm{B}(\kappa) \Leftrightarrow \mathrm{D}(\kappa) \Rightarrow \mathrm{C}(\kappa)$
	MA+¬CH	$\mathrm{A}(\kappa) \Leftrightarrow \mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa) \Leftrightarrow \mathrm{D}(\kappa)$
ω_2	precal	$\mathrm{A}(\kappa) \Leftrightarrow \mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa) \Leftrightarrow \mathrm{D}(\kappa)$
	¬precal	$\mathrm{A}(\kappa) \Rightarrow \mathrm{B}(\kappa) \Leftrightarrow \mathrm{C}(\kappa) \Leftrightarrow \mathrm{D}(\kappa)$

Note that for $\kappa = \omega_2$, the assumption that κ is a precaliber of measure algebras is necessary and sufficient for the equivalence $\mathrm{A}(\kappa) \Leftrightarrow \mathrm{C}(\kappa)$ (here we might as well replace ω_2 by any $\kappa \geq \omega_2$ of uncountable cofinality).
Under MA+¬CH, ω_1 is a precaliber of measure algebras but it is known that the latter assumption is not strong enough to ensure $\mathrm{C}(\omega_1) \Rightarrow \mathrm{A}(\omega_1)$, see [24] and [9]. The proof of the lemma given in section 4 explains why it is easier to deal with ω_2 than with ω_1 here.

3 Auxiliary results.

We explain in this section the terminology and notation that we use. We also mention some facts needed in the next sections.

Radon measures. In the sequel, we consider only finite measures, so we say that μ is a *Radon measure* on a (Hausdorff topological) space T if μ is a finite complete measure defined on a σ–algebra containing all Borel subsets of T, and μ is inner regular with respect to compact sets.
For a cardinal number κ, we denote by μ_κ the completion of the usual product measure on the Cantor cube $\{0,1\}^\kappa$. Kakutani's theorem ([12], 416T) implies that μ_κ is a Radon measure on $\{0,1\}^\kappa$. Basic properties of μ_κ can be found in [7] 1.15–1.16, [11], 254, [12], 416.
Recall that every set $B \subseteq \{0,1\}^\kappa$ belonging to the product σ–algebra is determined by coordinates in some countable set $I \subseteq \omega_1$, that is $B = \pi_I^{-1}\pi_I(B)$, where π_I denotes the projection from $\{0,1\}^\kappa$ onto $\{0,1\}^I$. Hence by Kakutani's theorem for every measurable set $A \subseteq \{0,1\}^\kappa$ there is a set B such that B depends on countably many coordinates, $B \subseteq A$ and $\mu_\kappa(A \setminus B) = 0$.

Maharam types Let μ be a finite measure with a domain Σ and $\mathfrak{A}$ be its measure algebra. For $A \in \Sigma$ we denote by $A^\bullet$ the corresponding element of $\mathfrak{A}$.
The *Maharam type* $\tau(\mathfrak{A})$ of μ can be defined as the density character of the metric space $(\mathfrak{A}, \rho)$, where $\rho(a,b) = \mu(a \triangle b)$. It is easy to check that $\tau(\mathfrak{A})$ can be defined also as

$$\tau(\mathfrak{A}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq \Sigma,\ \mathcal{C} \text{ is } \triangle\text{-dense in } \Sigma\},$$

where $\mathcal{C}$ is said to be $\triangle$–dense in Σ if for every $E \in \Sigma$ and every $\varepsilon > 0$ there is $C \in \mathcal{C}$ such that $\mu(E \triangle C) < \varepsilon$. Unless μ is atomic, we can also write

$\tau(\mu) = \dim L^1(\mu)$, i.e. $\tau(\mu)$ agrees with the density character of a Banach space $L^1(\mu)$.
Recall that a measure μ is *homogenous* if it has the same type on every $E \subseteq \Sigma$ with $\mu(E) > 0$. The essential part of the Maharam theorem (see [7], Theorem 3.5) states that if μ is a homogenous probability measure of type κ then its measure algebra $\mathfrak{A}$ is isomorphic to the measure algebra of μ_κ.
It is known that if a compact space K admits a Radon measure of type κ then for every $\tau \le \kappa$ one can define on K a Radon measure of type τ, see [13] and [22], Lemma 2.
Finally note that for every κ of uncountable cofinality, if a compact space K admits a Radon measure of type κ then it also admits a homogenous Radon measure of type κ.

Precalibers. A cardinal number κ is a *precaliber* of a Boolean algebra $\mathfrak{A}$ if for every $(a_\xi)_{\xi<\kappa} \subseteq \mathfrak{A}^+$ there is $X \subseteq \kappa$, $|X| = \kappa$, such that $(a_\xi)_{\xi \in X}$ is centered, i.e. $\prod_{\xi\in J} a_\xi \neq 0$ for every finite $J \subseteq X$.
We enclose here the following useful and known fact.

Lemma 3.1 *Let $\mathfrak{A}$ be the measure algebra of a Radon measure μ. A cardinal number κ is a precaliber of $\mathfrak{A}$ if and only if κ is a caliber of μ in the following sense:*
For every family $(E_\xi)_{\xi<\kappa}$ of measurable sets of positive measure there is $X \subseteq \kappa$ with $|X| = \kappa$, such that $\bigcap_{\xi\in X} E_\xi \neq \emptyset$.

In the sequel, we shall work under the assumption that a given κ is a precaliber of **all** measure algebras. It is worth noting that such a statement requires checking only one suitable measure algebra. The following fact, observed by D.H. Fremlin can be deduced from the Maharam theorem.

Lemma 3.2 *A cardinal number κ is a precaliber of all measure algebra if and only if κ is a precaliber of the measure algebra of the measure μ_κ.*

Some facts concerning precalibers of measure algebras can be found in [2], [3], [7], [8], A2U (see also [4]). Recall that $\mathfrak{c}^+$ is a precaliber of measure algebras, while it is undecidable in ZFC whether ω_1 and $\mathfrak{c}$ have this property.

Independent families. A family of disjoint pairs $((A^0_\alpha, A^1_\alpha))_{\alpha<\kappa}$ is said to be independent if for every $I \in [\kappa]^{<\omega}$ and $\phi : I \to \{0,1\}$ we have $\bigcap_{\alpha\in I} A_\alpha^{\phi(\alpha)} \neq \emptyset$.
In the case when all the sets A^i_α are measurable with respect to some measure μ, we say that a family of pairs $((A^0_\alpha, A^1_\alpha))_{\alpha<\kappa}$ is μ–independent if we have $\mu(\bigcap_{\alpha\in I} A_\alpha^{\phi(\alpha)}) > 0$ for every $I \in [\kappa]^{<\omega}$ and $\phi : I \to \{0,1\}$ (so μ–independence should not be confused with stochastic independence).
Independent families are the basic tool for defining continuous surjection onto Tychonoff cubes, as is explained in the following lemma (see [13], Lemma 1.1).

Lemma 3.3 *If K is a compact space and κ is a cardinal number then the following are equivalent*

(a) *there is a continuous surjection from K onto $[0,1]^\kappa$;*

(b) *there is an independent sequence $((F^0_\alpha, F^1_\alpha))_{\alpha<\kappa}$ of disjoint pairs, where F^0_α and F^1_α are closed subsets of K for every $\alpha < \kappa$.*

It is worth recalling that there is an inner topological characterization of compact spaces admitting a surjection onto a given Tychonoff cube, due to Shapirovskiĭ (see e.g. [26], Theorem 21).

Theorem 3.4 *The following are equivalent for a compact space K and an infinite cardinal κ:*

(i) *K can be continuously mapped onto $[0,1]^\kappa$;*

(ii) *there is a closed subspace F of K such that $\pi\chi(x,F) \geq \kappa$ for every $x \in F$.*

The number $\pi\chi(x,F)$ mentioned above denotes the π–character of a space F at a point x, i.e. minimal cardinality of a family $\mathcal{H}$ of nonempty open subsets of F such that every neighbourhood of x contains some member of $\mathcal{H}$.

Intersection numbers. The notion of intersection numbers introduced by Kelley [16] can be defined as follows.
Given a finite sequence $(P_1, \ldots, P_n)$ of sets, denote by $\mathrm{cal}(P_1, \ldots, P_n)$ the maximum of all numbers $|a|$, where $a \subseteq \{1, \ldots, n\}$ and $\bigcap_{i\in a} P_i \neq \emptyset$. Note that

$$\mathrm{cal}(P_1, \ldots, P_n) = \|\sum_{i=1}^{n} \chi_{P_i}\|,$$

where $\|\cdot\|$ denotes the supremum norm and χ_P is the characteristic function of a set P. The *intersection number* $\mathrm{k}(\mathcal{P})$ of an arbitrary family $\mathcal{P}$ of sets is defined as

$$\mathrm{k}(\mathcal{P}) = \inf\{\mathrm{cal}(P_1, \ldots, P_n)/n : \ P_i \in \mathcal{P}, n \geq 1\}.$$

It is a classical result due to Kelley [16], see also Chapter 6 in [4], that the condition $\mathrm{k}(\mathcal{P}) \geq r$ is equivalent to the existence of a finitely additive probability measure μ (defined on some algebra containing $\mathcal{P}$) with the property that $\mu(P) \geq r$ for all $P \in \mathcal{P}$. We shall need the following σ–additive version of this result, see [22], cf. [4], Lemma 6.3.

Theorem 3.5 *If $\mathcal{P}$ is a family of compact subsets of a topological space T then there exists a probability Radon measure μ such that $\mu(P) \geq k(\mathcal{P})$ for every $P \in \mathcal{P}$.*

4 The Fremlin–Haydon lemma.

In a Cantor cube $\{0,1\}^\kappa$ we denote by C^i_α, $\alpha<\kappa$, $i=0,1$ the one dimensional cylinders, i.e.

$$C^i_\alpha = \{x\in\{0,1\}^\kappa:\ x(\alpha)=i\}.$$

The lemma given below is due to R. Haydon [15] and was invented to provide a short argument for Theorem 5.1 below; the enclosed proof is derived from Fremlin [10]. Because of some further applications, it is written here in a somewhat complicated way, enabling us to state it in ZFC.

Lemma 4.1 *Let κ be any cardinal number such that $\kappa\ge\omega_2$. Suppose that $((A^0_\alpha,A^1_\alpha))_{\alpha<\kappa}$ is a sequence of pairs of measurable subsets of $\{0,1\}^\kappa$ with the following properties:*

(i) *$A^i_\alpha\subseteq C^i_\alpha$ for every $\alpha<\kappa$ and $i=0,1$;*

(ii) *$\mu_\kappa(A^0_\alpha)+\mu_\kappa(A^1_\alpha)>\frac12+r$ for every $\alpha<\kappa$;*

where r is a constant such that $0\le r<\frac12$. Then there are a sequence of measurable sets $(Z_\alpha)_{\alpha<\kappa}$ and $X\in[\kappa]^\kappa$ such that

(i) *$Z_\alpha\subseteq A^0_\alpha\cup A^1_\alpha$ and $\mu_\kappa(Z_\alpha)>2r$ for every $\alpha<\kappa$*

(ii) *for every $I\in[X]^{<\omega}$, if $\mu_\kappa(\bigcap_{\alpha\in I}Z_\alpha)>0$ then $((A^0_\alpha,A^1_\alpha))_{\alpha\in I}$ is a μ_κ–independent sequence.*

Proof. Recall that $(\{0,1\}^\kappa,\oplus)$ is a compact topological group if we denote by $\oplus$ the coordinatewise addition mod 2, and μ_κ is a Haar measure of that group. Let $s_\alpha:\{0,1\}^\kappa\to\{0,1\}^\kappa$ be a mapping defined by $s_\alpha(x)=x\oplus e_\alpha$, where $e_\alpha(\beta)=1$ iff $\alpha=\beta$. Then s_α is a measure preserving homeomorphism of $\{0,1\}^\kappa$.

For every $\alpha<\kappa$ we can find zero (i.e. closed and depending on countably many coordinates) sets $Z^i_\alpha\subseteq A^i_\alpha$, $i=0,1$, such that $\mu_\kappa(Z^0_\alpha)+\mu_\kappa(Z^1_\alpha)>\frac12+r$. Put $H_\alpha=Z^0_\alpha\cup Z^1_\alpha$ and $Z_\alpha=H_\alpha\cap s_\alpha[H_\alpha]$. Then

$$\mu_\kappa(Z_\alpha)=\mu_\kappa(H_\alpha\cap s_\alpha[H_\alpha])=2\mu_\kappa(H_\alpha)-\mu_\kappa(H_\alpha\cup s_\alpha[H_\alpha])>2r.$$

Moreover, by the definition of s_α, the set Z_α depends on coordinates in some countable set $J_\alpha\subseteq\kappa\setminus\{\alpha\}$.

By Hajnal's Free Set Theorem (44.3 in [6]), using the assumption $\kappa\ge\omega_2$ we can find a set $X\in[\kappa]^\kappa$ such that $\alpha\notin J_\beta$ whenever $\alpha,\beta\in X$.

For any finite $I\subseteq X$ and any function $\phi:I\to\{0,1\}$ we have

$$\mu_\kappa(\bigcap_{\alpha\in I}A^{\phi(\alpha)}_\alpha)\ge\mu_\kappa(\bigcap_{\alpha\in I}Z_\alpha\cap C^{\phi(\alpha)}_\alpha)=\frac{1}{2^{|I|}}\mu_\kappa(\bigcap_{\alpha\in I}Z_\alpha),$$

so we are done. ∎

It will be convenient to rewrite the basic lemma given above in a more general setting. The proof of the next result is a simple consequence of the fact that we are dealing with the properties that are preserved by isomorphisms of measure algebras.

Corollary 4.2 *Let μ be a homogenous Radon measure of type $\kappa \geq \omega_2$ on a compact space K. Then for every $0 \leq c < 1$ there are a sequence $((F^0_\alpha, F^1_\alpha))_{\alpha<\kappa}$ of pairs of disjoint closed subsets of K and a sequence $(G_\alpha)_{\alpha<\kappa}$ of Borel subsets of K such that*

(i) $G_\alpha \subseteq F^0_\alpha \cup F^1_\alpha$ *and* $\mu_\kappa(G_\alpha) > c$ *for every* $\alpha < \kappa$*;*

(ii) *for every* $I \in [\kappa]^{<\omega}$*, if* $\mu(\bigcap_{\alpha\in I} G_\alpha) > 0$ *then* $((F^0_\alpha, F^1_\alpha))_{\alpha\in I}$ *is* μ*–independent.*

Proof. Since μ is a homogenous measure of type κ, then there is an isomorpish $\theta : \mathfrak{A}_\kappa \to \mathfrak{A}$ between the measure algebra $\mathfrak{A}$ of μ and the measure algebra $\mathfrak{A}_\kappa$ of μ_κ. For every $\alpha < \kappa$ we may find a Borel set $B_\alpha \subseteq K$ such that $B_\alpha{}^\bullet = \theta(C^0_\alpha{}^\bullet)$. Next, we find closed sets F^0_α, F^1_α such that $F^0_\alpha \subseteq B_\alpha$, $F^1_\alpha \subseteq K \setminus B_\alpha$, and $\mu(F^0_\alpha) + \mu(F^1_\alpha) > \frac{1}{2} + c/2$, for every $\alpha < \kappa$.
We now apply Lemma 4.1 to $r = c/2$ and the family $((A^0_\alpha, A^1_\alpha))_{\alpha<\kappa}$, where A^i_α are chosen so that $A^i_\alpha \subseteq C^i_\alpha$, and $A^i_\alpha{}^\bullet = \theta^{-1}(F^i_\alpha{}^\bullet)$ for every $\alpha < \kappa$.
If X and Z_α are as in the assertion of Lemma 4.1 then we take Borel sets $G_\alpha \subseteq K$ such that $G_\alpha{}^\bullet = \theta(Z_\alpha{}^\bullet)$, for every $\alpha < \kappa$. It follows that $\mu(G_\alpha) > c$ for every $\alpha < \kappa$, and for every $I \in [X]^{<\omega}$, if $\mu(\bigcap_{\alpha\in I} G_\alpha) > 0$ then $((F^0_\alpha, F^1_\alpha))_{\alpha\in I}$ is a μ–independent sequence. ■

5 Three implications.

We are now ready to show how the basic lemma works.

Theorem 5.1 C(κ) $\Rightarrow$ A(κ) *whenever $\kappa \geq \omega_2$ is a precaliber of measure algebras.*

Proof. Let $\mu \in P(K)$ be homogenous of type $\kappa \geq \omega_2$. In the notation of Corollary 4.2, where $c > 0$, we have $\mu(G_\alpha) \geq c$ so we can find $X \in [\kappa]^\kappa$, such that the family $(G_\alpha{}^\bullet)_{\alpha\in X}$ is centered (in the measure algebra of μ). Then the family $((F^0_\alpha, F^1_\alpha))_{\alpha\in X}$ is independent by Corollary 4.2 and we get A(κ) applying Lemma 3.3. ■

Theorem 5.2 C(κ) $\Rightarrow$ D(κ) *for every $\kappa \geq \omega_2$.*

Proof. (1) We fix c with $1 > c > 1/2$ and, keeping the notation form Corollary 4.2, for every $\alpha < \kappa$ and $i = 0, 1$ we put

$$M^i_\alpha = \{\nu \in P(K) : \ \nu(F^i_\alpha) \geq c\}.$$

In this way for every α we have defined a disjoint pair (M^0_α, M^1_α) of closed subsets of $P(K)$, so the proof will be complete if we check that $((M^0_\alpha, M^1_\alpha))_{\alpha<\kappa}$ is an independent family.

(2) Take any finite $I \subseteq \kappa$ and a function $\phi : I \to \{0,1\}$; denote $H_\alpha = F_\alpha^{\phi(\alpha)}$ for simplicity. We want to check that $\bigcap_{\alpha\in I} M_\alpha^{\phi(\alpha)} \neq \emptyset$; in view of Theorem 3.5 it will do to check that $\mathrm{k}\{H_\alpha : \alpha \in I\} \geq c$, that is

$$||\sum_{\alpha\in I} n_\alpha \chi_{H_\alpha}|| \geq c \sum_{\alpha\in I} n_\alpha,$$

for any natural numbers n_α.
(3) For a function $f = \sum_{\alpha\in I} n_\alpha \chi_{G_\alpha}$ we have

$$\mathrm{esssup}(f) \geq \int f \,\mathrm{d}\mu \geq c \sum_{\alpha\in I} n_\alpha.$$

Writing

$$H = \{t \in K : f(t) = \mathrm{esssup}(f)\} \quad \text{and} \quad J = \{\alpha \in I : H \subseteq G_\alpha\},$$

we have $\mu(\bigcap_{\alpha\in J} G_\alpha) > 0$ so $\bigcap_{\alpha\in J} H_\alpha \neq \emptyset$. Taking any $t \in \bigcap_{\alpha\in J} H_\alpha$ we conclude that

$$||\sum_{\alpha\in I} n_\alpha \chi_{H_\alpha}|| \geq \sum_{\alpha\in I} n_\alpha \chi_{H_\alpha}(t) \geq \sum_{\alpha\in J} n_\alpha = \mathrm{esssup}(f) \geq c \sum_{\alpha\in I} n_\alpha. \blacksquare$$

Theorem 5.3 C$(\kappa) \Rightarrow$ B(κ) *for every* $\kappa \geq \omega_2$.

Proof. Again we keep the notation from Corollary 4.2, where $1 > c > 1/2$. For every $\xi < \kappa$ there is a continuous function $g_\xi : K \to [-1,1]$ such that $g_\xi(t) = 1$ for every $t \in F_\xi^0$ and $g_\xi(t) = -1$ for every $t \in F_\xi^1$. We shall check that the family $(g_\xi)_{\xi<\kappa} \subseteq C(K)$ is equivalent to the usual basis of $l^1(\kappa)$ with a constant $2c-1$, i.e. for any real numbers $r_i \in \mathbf{R}$ we have

$$||\sum_{i\leq n} r_i g_{\alpha_i}|| \geq (2c-1) \sum_{i\leq n} |r_i|.$$

Consider a Borel function $f = \sum_{i\leq n} |r_i| \chi_{G_{\alpha_i}}$; write

$$H = \{t \in K : f(t) = \mathrm{esssup} f\} \quad \text{and} \quad M = \{i \leq n : T \subseteq G_{\alpha_i}\}.$$

For $t \in H$ we have

$$f(t) \geq \sum_{i\in M} |r_i| = \mathrm{esssup}(f) \geq \int f \mathrm{d}\mu \geq c \sum_{i\leq n} |r_i|.$$

Taking $I = \{\alpha_i : i \in M\}$, and $\phi : I \to \{0,1\}$, where $\phi(\alpha_i) = 0$ iff $c_i \geq 0$, we get from *(ii)* of Corollary 4.2 that the set $A = \bigcap_{\alpha\in I} F_\alpha^{\phi(\alpha)}$ is nonempty. Taking any $t \in A$ we have

$$||\sum_{i\leq n} r_i g_{\alpha_i}(t)|| \geq \sum_{i\leq n} r_i g_{\alpha_i}(t) = \sum_{i\in M} r_i g_{\alpha_i}(t) + \sum_{i\notin M} r_i g_{\alpha_i}(t) \geq$$

$$\geq \sum_{i\in M} |r_i| - \sum_{i\notin M} |r_i| = 2 \sum_{i\in M} |r_i| - \sum_{i\leq n} |r_i| \geq (2c-1) \sum_{i\leq n} |r_i|,$$

and we are done. $\blacksquare$

6 When $A(\kappa) \Leftrightarrow C(\kappa)$.

By Theorem 5.1 and Theorem 4.2(a) from [24], if $\kappa \geq \omega_2$ has uncountable cofinality then $A(\kappa) \Leftrightarrow C(\kappa)$ holds if and only if κ is a precaliber of measure algebras. What about κ of countable cofinality? Note that if κ is a precaliber of measure algebras then necessarily $\mathrm{cf}(\kappa) > \omega$. R. Haydon suggested the following modification of the notion of precaliber.

Definition 6.1 *Say that a cardinal number κ is an m–precaliber of measure algebras if for every measure algebra $(\mathfrak{A}, \mu)$ and a family $(a_\xi)_{\xi<\kappa} \subseteq \mathfrak{A}$ with the property that $\inf_{\xi<\kappa} \mu(a_\xi) > 0$ there is $X \in [\kappa]^\kappa$ such that $(a_\xi)_{\xi \in X}$ is centered.*

Note that ω is an m–precaliber of measure algebras. Note also that for every κ of uncountable cofinality, κ is an m–precaliber if and only if it is a precaliber of measure algebras.
We now see that the proof of Theorem 5.1 above works under the assumption that κ is an m–precaliber. Using essentially the same idea as in [24] we get the following result.

Theorem 6.2 *If an infinite cardinal number κ is not an m-precaliber of measure algebras then there is a compact space K such that K carries a homogenous Radon measure of type κ but there is no continuous surjection from K onto $[0,1]^\kappa$.*

Proof. (1) Using Lemma 3.1 and 3.2 we can easily check that since κ is not an m–precaliber of measure algebras then we can find a constant $c > 0$ and a family $(B_\xi)_{\xi<\kappa}$ of measurable subsets of $\{0,1\}^\kappa$ such that $\mu_\kappa(B_\xi) > c$ for every $\xi < \kappa$ and there is no centered subfamily of $(B_\xi)_{\xi<\kappa}$ of size κ. Without loss of generality we can assume that every set B_ξ is compact.
Denoting one dimensional cylinders in $\{0,1\}^\kappa$ by C^i_ξ (see the beginning of section 4) we define compact sets $F_\xi \subseteq \{0,1\}^\kappa \times \{0,1\}^\kappa$ by $F_\xi = B_\xi \times C^0_\xi$ for every $\xi < \kappa$. Let μ be the product measure $\mu_\kappa \times \mu_\kappa$ on $\{0,1\}^\kappa \times \{0,1\}^\kappa$.
(2) We let $\mathcal{A}$ to be the algebra of sets generated by the family $\mathcal{F} = (F_\xi)_{\xi<\kappa}$, K to be its Stone space, i.e. K is a space of all ultrafilters in $\mathcal{A}$ with the basis consisting of the sets of the form $\widehat{A} = \{p \in K : A \in p\}$ for $A \in \mathcal{A}$. Then K is a compact space; we shall check that it has the required properties.
(3) Consider any closed subset H of K, and take a maximal subfamily $\mathcal{F}_0$ of $\mathcal{F}$ for which $\mathcal{H} = \{\widehat{F} \cap H : F \in \mathcal{F}_0\}$ is centered. By maximality and compactness $\bigcap \mathcal{H}$ consists of the unique point of H, say p.
No centered subfamily of $\mathcal{F}$ can have size κ so we have $|\mathcal{F}_0| < \kappa$. Since finite intersections of elements from $\mathcal{H}$ form a topological base of the space H at p it follows that the character of H at p is less than κ.
In particular, we have checked that there is no closed $H \subseteq K$ which would have π-character at least κ at every point, and by Shapirovskiĭ's theorem (see 3.4 above) K cannot be mapped onto $[0,1]^\kappa$.
(4) The measure μ restricted to the algebra $\mathcal{A}$ defines uniquely a Radon measure ν on K (by the formula $\nu(\widehat{A}) = \mu(A)$ for every $A \in \mathcal{A}$). It suffices

to check that there is a set $H \subseteq K$ such that $\nu(H) > 0$ and ν is homogenous of type κ on H.
Note first that $\mu(F_\xi \triangle F_\eta) \geq c/2$ whenever $\eta < \xi < \kappa$. Indeed,

$$\mu(F_\xi \triangle F_\eta) = \mu((B_\xi \times C_\xi^0) \setminus F_\eta)) + \mu(F_\eta \setminus (B_\xi \times C_\xi^0)) \geq$$

$$\geq \frac{1}{2}\mu((B_\xi \times \{0,1\}^\kappa) \setminus F_\eta) + \frac{1}{2}\mu((B_\xi \times \{0,1\}^\kappa) \cap F_\eta) = \frac{1}{2}\mu_\kappa(B_\xi) \geq \frac{c}{2}.$$

If we suppose that ν is nowhere homogenous of type κ then for every $\varepsilon > 0$ there is $H \subseteq K$ such that $\nu(H) > 1 - \varepsilon$ and ν is of type $< \kappa$ on H. But for H with $\nu(H) > 1 - c/4$ we have

$$\nu((\widehat{F_\xi} \cap H) \triangle (\widehat{F_\eta} \cap H)) = \nu((\widehat{F_\xi} \triangle \widehat{F_\eta}) \cap H) \geq \mu(F_\xi \triangle F_\eta) - c/4 \geq c/4,$$

for $\eta < \xi < \kappa$, a contradiction. ■

Corollary 6.3 *For an arbitrary cardinal number $\kappa \geq \omega_2$ the following are equivalent*

(i) $\mathrm{A}(\kappa) \Leftrightarrow \mathrm{C}(\kappa)$;

(ii) *κ is an m-precaliber of measure algebras.*

Proof. For $(ii) \to (i)$ we can repeat the proof of Theorem 5.1; the reverse implication follows from the Theorem 6.2. ■

For the sake of the above result it would be interesting to answer the following problem posed by R. Haydon (see [10]):

Problem 6.4 *Suppose that $\kappa = \sup_n \kappa_n$ where every κ_n is an m–precaliber of measure algebras. Does κ have the same property?*

7 Concluding remarks.

Recall that $\mathrm{C}(\kappa) \Rightarrow \mathrm{D}(\kappa)$ is undecidable in ZFC for $\kappa = \omega_1$. Indeed, assuming CH we can consider the space S constructed by Haydon in [14], Theorem 3.1. This space carries a homogenous Radon measure of type ω_1, and it follows from (v) of Theorem 3.1 that $P(S)$ is of cardinality ω_1, so $P(S)$ cannot be mapped onto $[0,1]^{\omega_1}$. On the other hand, under MA +nonCH we have $\mathrm{C}(\kappa) \Rightarrow \mathrm{A}(\kappa)$ so also $\mathrm{C}(\kappa) \Rightarrow \mathrm{D}(\kappa)$.
It follows from Theorem 5.2 that if the space $P(K)$ has tightness $\leq \omega_1$ then every Radon measure on K is of type $\leq \omega_1$.
The following problem has a positive solution under MA $+\neg$CH but perhaps can be answered in ZFC:

Problem 7.1 *Assume that $P(K)$ has countable tightness (or is even a Frechet space). Does every $\mu \in P(K)$ have a countable type?*

References

[1] S. Argyros, *On non–separable Banach spaces*, Trans. Amer. Math. Soc. 270 (1982), 193–216.

[2] J. Cichoń, A. Szymański, B. Węglorz, *On intersections of sets of positive Lebesgue measure*, Coll. Math. 52 (1987), 173–177.

[3] J. Cichoń, *On two cardinal properties of ideals*, Tran. Amer. Math. Soc. 314 (1989), 693–708.

[4] W.W. Comfort, S. Negrepontis, *Chain conditions in topology*, Cambridge Univ. Press 1982.

[5] M. Džamonja, K. Kunen, *Measures on compact HS spaces*, Fund. Math. 143 (1993), 41–54.

[6] P. Erdös, A. Hanjal, A. Máté, R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Akadémiai Kiadó, 1984.

[7] D.H. Fremlin, *Measure algebras*, in: Handbook of Boolean algebras, J.D. Monk (ed.), North–Holand 1989, Vol. III, Chap. 22.

[8] D.H. Fremlin, *Real–valued measurable cardinals*, Israel Math. Conf. Proceedings, vol. 6 (1993), 961–1044.

[9] D.H. Fremlin, *On compact spaces carrying Radon measures of uncountable Maharam type*, Fund. Math. 154 (1997), 295–304.

[10] D.H. Fremlin, *On Haydon's property*, note of 21.9.00.

[11] D.H. Fremlin, *Measure Theory, Vol. 2: Broad Foundations,* Torres Fremlin (Colchester), 2001.

[12] D.H. Fremlin, *Measure Theory, Vol. 4: Topological measure spaces,* in preparation. *Partial drafts available via*

`http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`.

[13] R. Haydon, *On Banach spaces which contain $l^1(\tau)$ and types of measures on compact spaces*, Israel J. Math. 28 (1977) 313–324.

[14] R. Haydon, *On dual L^1–spaces and injective bidual Banach spaces*, Israel J. Math. 31 (1978) 142–152.

[15] R. Haydon, unpublished.

[16] J.L. Kelley, *Measures on Boolean algebras*, Pacific J. Math. 9 (1959), 1165–1177.

[17] K. Kunen, *A compact L–space under CH*, Topology Appl. 12 (1981), 283–287.

[18] K. Kunen, J. van Mill, *Measures on Corson compact spaces*, Fund. Math. 147 (1995), 61–72.

[19] S. Negrepontis, *Banach spaces and topology*, in: Open problems in topology, J. van Mill, G.M. Reed (edts.), Chapter 23.

[20] A. Pełczyński, *On Banach spaces containing* $L^1(\mu)$, Studia Math. 30 (1968), 231–246.

[21] G. Plebanek, *Compact spaces that result from adequate families of sets,* Topology and Appl. 65 (1995), 257–270.

[22] G. Plebanek, *A note on strictly positive Radon measures*, Colloq. Math. 69 (1995), 87–92.

[23] G. Plebanek, *On Radon measures on first–countable spaces*, Fund. Math. 148 (1995), 159–164.

[24] G. Plebanek, *Nonseparable Radon measures and small compact spaces*, Fund. Math. 153 (1997), 25–40.

[25] Z. Semadeni, *Banach spaces of continuous functions*, (PWN, Warszawa, 1971).

[26] B.E. Shapirovskiĭ, *Special types of embedding in Tychonoff cubes. Subspaces of* Σ*–products and cardinal invariants*, in: A. Csaszar, ed., Topology vol. II (North–Holand, Amsterdam 1980) 1055–186.

[27] M. Talagrand, *Separabilité vague dans l'espace des mesures sur un compact*, Israel J. Math. 37 (1980), 171–180.

[28] M. Talagrand, *Sur les espace de Banach contenant* l^τ , Israel J. Math. 40 (1981), 324–330.

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