Measures, densisties and some compactifications of ω

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Hejnice 2008

Basic motivation

Is there a Banach space E, possibly in the form $E = C(K)$, which has the Mazur property but does not have the Gelfand–Phillips property?

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A problem

Is there a compactification K of ω such that

- no subsequence of ω is convergent in K;
- every measure on K can be approached sequentially by measures concentrated on ω?

Basic terminology & notation

If $\mathfrak{A}$ is a Boolean algebra then $P(\mathfrak{A})$ denotes the set of all finitely additive probability measures on $\mathfrak{A}$.
For K compact zerodim., $P(K) = P(\mathrm{Clopen(K)})$.
Consider a topology on $P(\mathfrak{A})$ generated by the mappings $\mu \to \mu(a)$ for $a \in \mathfrak{A}$; for instance $\mu_n \to \mu$ means that $\mu_n(a) \to \mu(a)$ for every $a \in \mathfrak{A}$. Then $P(\mathfrak{A})$ is a compact space.

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Sequential closure

If K is compact and zerodim., and $X \subseteq K$ then write

- $\mathrm{conv}(X)$ for all $\mu \in P(K)$ which are finite combinations of Dirac measures δ_x, $x \in X$;
- $S_1(X)$ for all the limit of converging sequences from $\mathrm{conv}(X)$;
- $S(X) = \bigcup_{\xi<\omega_1} S_\xi(X)$ for the least sequentially closed set of measures containing $\mathrm{conv}(X)$.

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Remark

Checking $S(\omega) = P(K)$ may be done in two steps:

- $\delta_t \in S(\omega)$ for $t \in K \setminus \omega$;
- $S(K) = P(K)$?

Minimally generated algebras

$\mathfrak{A}$ is minimally generated if $\mathfrak{A} = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$, where $\mathfrak{A}_\xi$ form an increasing continuous chain, $\mathfrak{A}_0 = \{0,1\}$, $\mathfrak{A}_{\xi+1}$ is a minimal extension of $\mathfrak{A}_\xi$ for $\xi < \alpha$.
Introduced by Koppelberg; investigated by Fedorčuk, Shelah, Dow, Koszmider. Related papers: Borodulin-Nadzieja [2007], Džamonja & GP [2007]

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When $S(K) = P(K)$

Let $\mathfrak{A} = \mathrm{Clopen(K)}$ be minimally generated.

- If $\alpha = \omega_1$ then $S_1(K) = P(K)$; moreover
- under $\diamondsuit$, there is K such that K contains no converging sequences but every **nonatomic** $\mu \in P(K)$ is G_δ (Džamonja & GP).
- $S(K) = P(K)$; cf. Borodulin-Nadzieja [2007].

More on $S(K) = P(K)$

If $K = \{0,1\}^{\mathfrak{c}}$ then $S_1(K) = P(K)$, see

- Losert [1979] under CH;
- Frankiewicz & GP [1995] under MA;
- Fremlin [20??] in ZFC.

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More on $S(K) = P(K)$

It is rel. consistent that $S_1(K) = P(K)$ for ever compact space K such that $\chi(x,K) < \mathfrak{c}$ for every $x \in K$ (GP [2000]); cf. Mercourakis [1996].

Back to our problem

Recall that we look for a compactification $\omega \subseteq K$, such that ω has no converging subsequences, and $S(\omega) = P(K)$. Suppose that we construct K as $\mathrm{ULT}(\mathfrak{A})$ for some minimally generated $\mathfrak{A}$ on ω. We shall get $S(\omega) = P(K)$ if we ensure that

$$\delta_x \in S(\omega) \text{ for } x \in K \setminus \omega.$$

Back to our problem

Recall that we look for a compactification $\omega \subseteq K$, such that ω has no converging subsequences, and $S(\omega) = P(K)$. Suppose that we construct K as $\mathrm{ULT}(\mathfrak{A})$ for some minimally generated $\mathfrak{A}$ on ω. We shall get $S(\omega) = P(K)$ if we ensure that

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A partial example

For $A \subseteq \omega$ write $d(A) = \lim_{n\to\infty} |A \cap n|/n$.
Suppose that the ideal of sets of density zero contains base A_ξ, $\xi < \omega_1$, where $A_\xi \subseteq^* A_{\xi+1}$ for every ξ. Let $\mathfrak{A}$ be generated by all A_ξ and finite sets, and let $K = \mathrm{ULT}(\mathfrak{A})$.
Consider $p \in K$ containing all $\omega \setminus A_\xi$. Then no sequence of ω converges to p. But $S_1(\omega) = P(\omega)$, in particular

$$\delta_p = \lim_{n\to\infty} 1/n \sum_{i<n} \delta_i.$$

Can we use Balcar-Pelant-Simon tree?

If $\mathcal{T}$ is a π–base of infinite subsets of ω, such that $\mathcal{T}$ is a $\subseteq^*$ tree of height $\mathfrak{h}$, then $\mathcal{T}$ generates a minimally generated algebra.

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A question

Is it consistent, that for every $\subseteq^*$–decreasing sequence $A_\xi \subseteq \omega$, $\xi < \mathfrak{h}$ there is some density d such that $d(A_\xi) = 1$ for $\xi < \mathfrak{h}$?
Here a density may be defined as

$$d_X(A) = \lim_n \frac{|A \cap X \cap n|}{|X \cap n|} \text{ for some } X \subseteq \omega, \text{ or}$$

$$d_g(A) = \lim_n \frac{\sum_{i \in n \cap A} g(i)}{\sum_{i<n} g(i)} \text{ for some } g : \omega \to \mathbb{R}_+, \text{ or}$$

limits (of limits of limits...) of such.