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Independent families in measure algebras

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Definition. *A family* $\{a_\xi : \xi < \kappa\}$ *in a Boolean algebra* $\mathfrak{A}$ *is*

(i) **centred** *if* $\bigwedge_{\xi \in I} a_\xi \neq 0$ *for every finite* $I \subseteq \kappa$*;*

(ii) **independent** *if* $\bigwedge_{\xi \in I} a_\xi^{\phi(\xi)} \neq 0$ *for every finite* $I \subseteq \kappa$ *and every* $\phi : I \to \{0, 1\}$*.*

Definition. *A cardinal number* κ *is a* **precalibre** *of a Boolean algebra* $\mathfrak{A}$ *if for every family* $\{a_\xi : \xi < \kappa\} \subseteq \mathfrak{A}^+$ *there is* $X \in [\kappa]^\kappa$ *such that* $\{a_\xi : \xi \in X\}$ *is centred.*

Definition. *A cardinal number* κ *is an* **independence precalibre** *of a Boolean algebra* $\mathfrak{A}$ *if for every family* $\{a_\xi : \xi < \kappa\}$ *of distinct elements of* $\mathfrak{A}$ *there is* $X \in [\kappa]^\kappa$ *such that* $\{a_\xi : \xi \in X\}$ *is independent.*

Definition. *For a measure algebra* $(\mathfrak{A}, \mu)$ *a cardinal number* κ *is a* **measure precalibre** *if in every family* $\{a_\xi : \xi < \kappa\} \subseteq \mathfrak{A}$ *with* $\inf_{\xi<\kappa} \mu(a_\xi) > 0$ *there is a centred subfamily of size* κ.

Definition. *For a measure algebra* $(\mathfrak{A}, \mu)$

(i) a family $\{a_\xi : \xi < \kappa\} \subseteq \mathfrak{A}$ *is* **separated** *if there is* $\varepsilon > 0$ *such that* $\mu(a_\xi \triangle a_\eta) \geq \varepsilon$ *for* $\xi \neq \eta$;

(ii) a cardinal number κ *is a* **measure independence precalibre** *of* $\mathfrak{A}$ *if every separated family* $\{a_\xi : \xi < \kappa\} \subseteq \mathfrak{A}$ *contains an independent subfamily of size* κ.

Definition. *κ is (*)–precalibre of measure algebras if for every $(\mathfrak{A}, \mu)$ and $\{a_\xi : \xi < \kappa\} \subseteq \mathfrak{A}$ satisfying (A) there is $X \in [\kappa]^\kappa$ such that $\{a_\xi : \xi \in X\}$ has property (P), where*

(*)	*(A)*	*(P)*
precal.	$a_\xi \neq 0$	*centred*
ind. precal.	$a_\xi \neq a_\eta$	*independent*
m. precal.	$\mu(a_\xi) \geq \varepsilon$	*centred*
m. ind. precal.	$\mu(a_\xi \triangle a_\eta) \geq \varepsilon$	*independent*

Remarks.

(1) If $\mathrm{cof}(\kappa) > \omega$ then κ is a measure precalibre iff κ is a precalibre (of measure algebras).

(2) Every measure independence precalibre of measure algebras is a measure precalibre.

(3) No $\kappa \leq \mathbf{c}$ is independence precalibre of measure algebras.

Measure precalibres.

(1) ω is a measure precalibre.

(2) $\kappa < \mathbf{c}$ may be (under MA +non CH) and may not be (under CH) a measure precalibre; Fremlin vol. 5 of Measure Theory or Džamonja & G.P. [04].

(3) Problem. (Haydon) *Let* κ_n *be regular and precalibre of measure algebras. Is* $\kappa = \sup_{n<\omega} \kappa_n$ *a measure precalibre?*

(4) Problem. (Fremlin) *Is it rel. consistent that every regular* κ *is a precalibre of measure algebras?*

Fact. (Shelah, Argyros & Tsarpalias, Fremlin) *If* $\mathrm{cof}(\kappa) = \omega$ *and* $2^\kappa = \kappa^+$ *then* κ^+ *is not a precalibre of measure algebras.*

Measure independence precalibres.

(1) Fact. *ω is a measure independence precalibre.*

(2) Theorem. (Argyros & Tsarpalias [82]) *Assume that $\kappa = \mathrm{cof}(\kappa)$ and $\tau^\omega < \kappa$ for $\tau < \kappa$ (for instance: $\kappa = \mathbf{c}^+$. Then κ is an independence precalibre of all* ccc *Boolean algebras, so in particular of all measure algebras* **(Haydon [77])**.

(3) Theorem. (Shelah [99]; Džamonja & G.P. [04]) *Suppose that for some θ,*

$$\theta = \theta^\omega < \mathrm{cof}(\kappa) \le \kappa \le 2^\theta.$$

Then κ is a measure independence precalibre. If, moreover, $\kappa > 2^{\mathbf{c}}$ then κ is an independence precalibre of measure algebras.

Theorem. (Fremlin & G.P.)
For $\kappa \geq \omega_2$ TFAE

(i) κ is a measure precalibre;

(ii) κ is a measure independence precalibre.

Theorem. (Fremlin [97]) *Under $MA + nonCH$, ω_1 is a measure independence precalibre.*

Theorem. (G.P. [97]) *It is rel. consistent that ω_1 is a measure precalibre but not measure independence precalibre.*

About the proof.

Theorem. (Hajnal's free set theorem) *If $\kappa \geq \omega_2$ and $J : \kappa \to [\kappa]^{\leq\omega}$ is a set mapping such that $\xi \notin J_\xi$ for every $\xi < \kappa$ then there is $X \in [\kappa]^\kappa$ such that $\eta \notin J_\xi$ for all $\eta, \xi \in X$.*

(1) Lemma. *Let $\kappa \geq \omega_2$ have uncountable cofinality. If $\{s_\xi : \xi < \kappa\} \subseteq [\kappa]^{<\omega}$ is a pairwise disjoint family, $\{J_\xi : \xi < \kappa\} \subseteq [\kappa]^{\leq\omega}$ are such that $s_\xi \cap J_\xi = \emptyset$ for every $\xi < \kappa$ then there is $X \subseteq \kappa$ of cardinality κ such that $s_\xi \cap J_\eta = \emptyset$ whenever $\xi, \eta \in X$.*

(2) Lemma. *Let $B \subseteq \{0,1\}^\kappa$ be a measurable set and $X \subseteq \kappa$ be such that $B^\bullet \notin \mathfrak{A}[X]$. Then there are a finite set $s \subseteq \kappa \setminus X$, a countable set $J \subseteq \kappa \setminus s$, nonempty clopen sets $C(0), C(1) \sim s$, a set $Z \sim J$ with $\lambda(Z) > 0$ such that $Z \cap C(i) \subseteq B^i$ for $i = 0, 1$.*

Consider a separated family $\{B_\xi : \xi < \kappa\}$ of subsets of $\{0,1\}^\kappa$. Let $B_\xi \sim I_\xi$, $I_\xi \in [\kappa]^\omega$.
We can assume that

$$B_\xi^\bullet \notin \mathfrak{A}[X_\xi], \quad \text{where} \quad X_\xi = \bigcup_{\eta<\xi} J_\eta.$$

Apply Lemma 2 to every $B_\xi \notin \mathfrak{A}[X_\xi]$: there are pairwise disjoint finite sets s_ξ in κ, nonempty clopen sets $C_\xi(0), C_\xi(1) \sim s_\xi$, and sets Z_ξ of positive measure, where every $Z_\xi \sim J_\xi \subseteq \kappa \setminus s_\xi$.
We can assume that
$J_\xi \cap s_\eta = \emptyset$ for all ξ, η;
$\{Z_\xi^\bullet : \xi < \kappa\}$ is centred in $\mathfrak{A}$.

For any finite set $a \subseteq \kappa$ and $\varphi : a \to \{0,1\}$

$$\lambda(\bigcap_{\xi\in a} B_\xi^{\varphi(\xi)}) \geq \lambda(\bigcap_{\xi\in a} Z_\xi \cap C_\xi(\varphi(\xi))) =$$

$$= \lambda(\bigcap_{\xi\in a} Z_\xi) \cdot \prod_{\xi\in a} \lambda(C_\xi(\varphi(\xi))) > 0.$$

Haydon's property. Say that κ has Haydon's property if every compact space K that carries a Radon measures of Maharam type κ can be continously mapped onto $[0,1]^\kappa$.

Theorem. (G.P. [97] + Haydon + Fremlin) *For $\kappa \geq \omega_2$, TFAE*

(a) κ has Haydon's property;

(b) κ is a measure precalibre.

(c) κ is a measure independence precalibre.

Theorem. (Fremlin [97]) *Under MA + non CH, ω_1 has Haydon's property and (therefore) is a measure independence precalibre.*

Theorem. (G.P. [97]) *Let $\mathcal{N}$ be the null ideal of the measure on $\{0,1\}^{\omega_1}$. Suppose that* $\text{cov}(\mathcal{N}) > \omega_1$ *but there is a family $\{N_\xi : \xi < \omega_1\} \subseteq \mathcal{N}$ such that $\bigcup_{\xi<\omega_1} N_\xi$ meets every perfect set in $\{0,1\}^{\omega_1}$. Then ω_1 is a measure precalibre but does not have Haydon's property and (therefore) is not a measure independence precalibre.*

Theorem. (Kunen & van Mill [95] for $(b) \to (a)$; G.P. [95] for $(a) \to (b)$) *TFAE*

(a) ω_1 is a measure precalibre;

(b) every Radon measure on a first–countable (compact) space is of countable type.

Problem. *Can we replace in (b) "first–countable" by "countably tight"?*

Problem. *Suppose that* ω_1 *is a measure pre-calibre; let* $\{a_\xi : \xi < \omega_1\}$ *be a separated family in a measure algebra. Can we find* $X \in [\omega_1]^{\omega_1}$ *such that* $\{a_\xi : \xi \in X\}$ *is free, i.e.* $\bigwedge_{\xi \in I} a_\xi \wedge \bigwedge_{\xi \in J} a_\xi^c \neq 0$ *for every finite* $I, J \subseteq X$ *with* $\max I < \min J$*?*