What isomorphisms between $C(K)$ spaces cannot forget

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Stable properties

Let $\mathcal{P}$ be a class of compact spaces. Say that $\mathcal{P}$ is **stable** if for every $K \in \mathcal{P}$ and every compact L,

$$C(K) \simeq C(L) \Rightarrow L \in \mathcal{P}.$$

Here $C(K) \simeq C(L)$ denotes that Banach spaces $C(K), C(L)$ are isomorphic as Banach spaces (of continuous functions with the supremum norm).

Examples of stable classes/properties

- **Metrizable spaces**
  K is metrizable iff $C(K)$ is separable.
- **Eberlein compacta**
  K is Eberlein compact iff $C(K)$ is WCG.
- *ccc* **spaces**
  K is ccc iff $C(K)$ does not contain $c_0(\omega_1)$, Rosenthal [1969].
- **Spaces with a strictly positive measure**
  K carries a strictly positive measure iff $C(K)^*$ contains weak compact total subset, again Rosenthal, cf. Todorcevic [2000].
- **Rosenthal compacta**
  See below.

Unstable properties

- Separability is not a stable property:

$$C(\beta\omega) = l_\infty \simeq L_\infty[0,1] = C(S),$$

  S = the Stone space of the measure algebra of λ on $[0,1]$.
- By Miljutin's theorem, $C(2^\omega) \simeq C[0,1] \simeq C[0,1]^2$: conectedness and dimension are not stable.

Notation

- $C(K)^* = M(K)$;
- $P(K) \subseteq M(K)$; $M_1(K) \subseteq M(K)$;
- $t \in K$, $\delta_t \in P(K)$ is the Dirac measure.

If $T : C(K) \to C(L)$ then $T^* : M(L) \to M(K)$, where for $\nu \in M(L)$, $T^*\nu$ is defined by $T^*\nu(f) = \nu(Tf)$.

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Stability and spaces of measures

If $T : C(K) \to C(L)$ is an isomorphism then $T^* : M(L) \to M(K)$, T^* sends $\{\delta_t : t \in L\} = L$ to a bounded subset of $M(K)$.
Conclusion. A class $\mathcal{P}$ is stable provided

1. $K \in \mathcal{P}$, $L = \overline{L} \subseteq K \Rightarrow L \in \mathcal{P}$,
2. $K \in \mathcal{P} \Rightarrow M_1(K) \in \mathcal{P}$.

Example. Rosenthal compact spaces, see Godefroy [1980]

Corson compacta and first–countable spaces

K is Corson compact if for some κ

$$K \hookrightarrow \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$

REMARKS.

- Under CH, there are "pathological" first–countable Corson compacta K (Haydon, Kunen, Talagrand ...).
- there is such K of size $\mathfrak{c}$ with $|P(K)| = 2^{\mathfrak{c}}$, Fremlin & GP [2003].
- Under MA + non CH, Corson compacta behave properly.
- Consistently, $M_1(K)$ is first–countable if (and only if) K is first–countable, GP [2000].
- Under MA + non CH, first–countability is still unclear.

Sometimes stable

- Under MA + non CH, if K is Corson compact then $M_1(K)$ is Corson compact, see AMN [1988] and then Corson compacta form a stable class.
- Consistently, first–countability is stable GP [2000].
- For $\kappa \geq \omega$, $\mathcal{P}_\kappa =$ the class of spaces admitting surjection onto $[0,1]^\kappa$. Then $\mathcal{P}_\omega$ is stable; for every κ, it is consistent that $\mathcal{P}_\kappa$ is stable, Fremlin [1997], GP [1997].

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Problems

Assume CH. Show that

1. Corson compactness not stable,
2. first–countability not stable,
3. the class $\mathcal{P}_{\omega_1}$ is not stable.

Upper bound

If K is either Corson compact or first–countable then $C(K)$ has the Mazur property, i.e. every *weak** sequentially continuous φ on $M(K)$ is defined by some element of $C(K)$, GP [1993]. In particular, for any L with $C(L) \simeq C(K)$, $C(L)$ cannot contain l_∞ or $C[0, \omega_1]$

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Some results

Under CH,

1. there is first–countable Corson compact K, and a surjection $T : C(K) \to l_\infty$; in particular, $\beta\omega \hookrightarrow M_1(K)$;
2. there is first–countable Corson compact K, $L = \beta\omega \oplus L'$ and $T : C(K) \to C(L)$ which is 1–1 and has a dense image.

More than a conjecture

Under CH, the class of Corson compact spaces is **not** stable:

There is a first–countable Corson compact K, and a compact L containing the split interval, such that $C(L) \simeq C(K)$.