A Little Ado about Rectangles

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Abstract. We discuss a problem of Ulam: whether every subset of the plane can be obtained by making countably many set operations with generalized rectangles.

1. ULAM'S PROBLEM. In the 1930s, Stanisław Ulam asked a question, which was recorded in *The Scottish Book* [**15**] as Problem 99; it may be stated as follows:

Problem 1. Does every subset of $\mathbb{R} \times \mathbb{R}$ belong to the σ-algebra generated by the sets of the form $A \times B$ where A, B are arbitrary subsets of $\mathbb{R}$?

We call a *rectangle* any set in the plane that can be written as $A \times B$. The σ-algebra generated by the sets of the form $(a, b) \times (c, d)$ is the Borel σ-algebra of $\mathbb{R}^2$, so Ulam's question is really about arbitrary rectangles. Consider the following example.

Example 2. If $D \subseteq \mathbb{R}$ is a subset and $f : D \to \mathbb{R}$ is any function, then its graph $G_f = \{(x, f(x)) : x \in D\}$ can be written as

$$G_f = \bigcap_{n=1}^{\infty} \bigcup_{q \in \mathbb{Q}} f^{-1}\big([q - 1/n, q]\big) \times [q - 1/n, q],$$

since if (x, y) belongs to the right-hand side, then we conclude that $|f(x) - y| \leq 1/n$ for every n, and hence, $y = f(x)$. Therefore, the graph of every function belongs to the σ-algebra generated by rectangles.

It turned out that the answer to Problem 1 is neither positive nor negative; it cannot be settled within the usual axioms of set theory. In other words, it is impossible to produce either a proof that the answer is positive or a proof that it is negative. In this note, we explain the relation of this question to other better-known axioms and also to a problem in functional analysis. All the results presented are well known, and our aim is only to present them in an attractive way at an elementary level.

2. SOME SET THEORY. We say that the cardinality of a set X is smaller than or equal to the cardinality of a set Y, and we write $|X| \leq |Y|$, if there exists an injective function $f : X \longrightarrow Y$. The Cantor–Bernstein theorem asserts that if $|X| \leq |Y|$ and $|Y| \leq |X|$, then there is a bijection between X and Y. In such a case, we say that X and Y have the same cardinality, and we write $|Y| = |X|$. A consequence of the axiom of choice is that for every X and Y, either $|X| \leq |Y|$ or $|Y| \leq |X|$. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$ and called the *continuum*.

A linear order on a set X is called a well order if every nonempty subset of X has a least element. The typical example is the order of the natural numbers. A well order on an uncountable subset is harder to imagine, but a consequence of the axiom of choice is the following.

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Theorem 1 (Principle of well order). *For every set X, there exists a well-order relation $\prec$ on X.*

Given a set X, $x \in X$, and a well order $\prec$ on X, let $I_\prec(x) = \{t \in X : t \prec x\}$ be the initial segment of elements that are smaller than x.

Theorem 2. *For every set X, there exists a well-order relation on X such that $|I_\prec(x)| < |X|$ for all $x \in X$.*

Proof. Take any well order $\prec$ of X. If $\prec$ does not have the required property, then take y to be the minimum of all x such that $|I_\prec(x)| \geq |X|$. Then there is a injective function $f : X \longrightarrow I_\prec(y)$, and we can define a new order $\prec_1$ on X declaring that $x \prec_1 x'$ if and only if $f(x) \prec f(x')$. Then $\prec_1$ is as desired. ■

3. A POSITIVE ANSWER: THE CONTINUUM HYPOTHESIS. The most famous statement in mathematics that cannot be proved or disproved from the usual axioms is the following.

Axiom 3 (The Continuum Hypothesis). For every infinite set $A \subset \mathbb{R}$, either $|A| = |\mathbb{N}|$ or $|A| = \mathfrak{c}$.

The continuum hypothesis (CH for short) was stated by Cantor and constituted the first problem on Hilbert's famous list. Gödel [**8**] showed that it cannot be disproved and Cohen [**5, 6**] that it cannot be proved; see also [**9**] for the history of Hilbert's list.

We are going to show now that CH implies that Ulam's problem has a positive solution. This was first shown by Kunen [**13**] and Rao [**16**]. Take an economical well order $\prec$ of $\mathbb{R}$ as in Theorem 2, and note that under CH every initial segment $I_\prec(x)$ is countable. Consider the set D in the plane, where

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \prec x\}.$$

If S is any subset of $\mathbb{R} \times \mathbb{R}$ and $x, y \in \mathbb{R}$, then the vertical section S_x and horizontal section S^y are defined as $S_x = \{y \in \mathbb{R} : (x, y) \in S\}$, $S^y = \{x \in \mathbb{R} : (x, y) \in S\}$.

Take any set $S \subseteq D$. Then $S_x \subseteq I_\prec(x)$, so S_x is countable for every x. Therefore, if we write $A = \{x \in \mathbb{R} : S_x \neq \emptyset\}$, then we can enumerate S_x by a sequence $y_1(x), y_2(x), \ldots$. (Note that if S_x is finite, then we can repeat its elements.) In such a way, we have defined functions $y_n : A \longrightarrow \mathbb{R}$, and S is the union of their graphs. The graph of every function y_n can be written as in Example 2, and we conclude that S can be expressed in terms of rectangles using countably many operations.

This is half of the proof, but the other half is symmetric: Let S be any subset of $\mathbb{R} \times \mathbb{R}$. Consider the set C above the diagonal, that is, $C = \{(x, y) : x \prec y\}$. A similar argument as above interchanging the roles of the two variables shows that $S \cap C$ belongs to the σ-algebra generated by rectangles. Finally, if $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$, then $S \cap \Delta$ is the graph of a function, so we get that $S = (S \cap D) \cup (S \cap \Delta) \cup (S \cap C)$ belongs to the σ-algebra generated by rectangles.

Remark 4. It is possible to obtain the same conclusion from a weaker axiom known as Martin's Axiom, or even from $\mathfrak{p} = \mathfrak{c}$. See [**13**] and 21G in [**7**].

4. A NEGATIVE ANSWER: EXISTENCE OF MEASURES. The first time that we usually encounter σ-algebras is when we learn that Lebesgue measure λ does not measure *all* subsets of $\mathbb{R}$ but only sets from a certain σ-algebra, the σ-algebra of Lebesgue-measurable sets. Can we do better?

Problem 5. Is it possible to define a measure $\overline{\lambda}$ on all subsets of $\mathbb{R}$ that coincides with Lebesgue measure on the measurable sets?

This question was posed by Banach [**2**] after Vitali [**20**] had shown that no such $\overline{\lambda}$ exists that is invariant under translations. Banach and Kuratowski [**2**] showed that the continuum hypothesis implies that the answer to Problem 5 is negative. Therefore, it is impossible to prove that the answer to Problem 5 is positive. On the other hand, [**11, 17, 19**]:

It is impossible to prove that it is impossible to prove[1] *that the answer to Problem 5 is negative.*

Experts agree on the unprovable belief that the answer to Problem 5 cannot be proven false. Thus, the following assertion is now seen as a legitimate axiom.[2]

Axiom 6 (Full Measure Extension Axiom). There exists a measure $\overline{\lambda}$ defined on all subsets of $\mathbb{R}$ that coincides with Lebesgue measure on the measurable sets.

Let us check that this axiom implies a negative answer to Ulam's question. Let $\prec$ be a well order on $\mathbb{R}$ as in Theorem 1. If every initial segment $I_\prec(x)$ of this well order has measure zero, then set $X = \mathbb{R}$. Otherwise, take the first element x such that $\overline{\lambda}(I_\prec(x)) > 0$, and set $X = I_\prec(x)$. In either case, we then have a set X of positive measure with all its initial segments having measure zero. It's time to move to the plane—here, we have the product measure $\overline{\lambda} \otimes \overline{\lambda}$, which is defined on the σ-algebra $\mathcal{R}$ generated by all rectangles. Recall that the product measure satisfies the formula $\overline{\lambda} \otimes \overline{\lambda}(A \times B) = \overline{\lambda}(A) \cdot \overline{\lambda}(B)$ for all rectangles, and consequently, we have the following Fubini formula

$$\overline{\lambda} \otimes \overline{\lambda}(S) = \int_{-\infty}^{+\infty} \overline{\lambda}(S_x)\, \mathrm{d}\overline{\lambda}(x) = \int_{-\infty}^{+\infty} \overline{\lambda}(S^y)\, \mathrm{d}\overline{\lambda}(y),$$

for all S from $\mathcal{R}$. Take $D = \{(x, y) \in X^2 : y \prec x\}$, and note that $\overline{\lambda}(D_x) = 0$ for every vertical section since D_x is a initial segment of X and hence is of measure zero. On the other hand, $\overline{\lambda}(D^y) = \overline{\lambda}(X \setminus I_\prec(y)) = \overline{\lambda}(X) > 0$ for every horizontal section of D and every $y \in X$. Therefore, if we could calculate the measure of D at all, we would get $\overline{\lambda} \otimes \overline{\lambda}(D) = 0$ and $\overline{\lambda} \otimes \overline{\lambda}(D) = \overline{\lambda}(X) > 0$ at the same time! So the conclusion is that D is not in the σ-algebra generated by rectangles.

Remark 7. The measure-theoretic argument presented here is due to Kunen [**13**]; see also Rao [**16**]. In fact, for this argument, one does not really need the full strength of Axiom 6. The following weaker version would be enough.

Axiom 8 (Partial Measure Extension Axiom). For every countable family $\mathcal{A}$ of subsets of $\mathbb{R}$, there is an extension of Lebesgue measure to a measure $\overline{\lambda}$ on a σ-algebra containing $\mathcal{A}$.

[1] This repetition is not a typo.

[2] In fact, the legitimacy of the usual axioms of set theory (the so-called ZFC system) is also based on the consensus of the community because Gödel's second incompleteness theorem implies that we cannot prove that those axioms are not contradictory.

The advantage of this axiom is that one can actually prove that neither it nor its negation can be proven from the usual axioms of set theory, a result due to Carlson [**4, 10**]. So from this, we can conclude that it is impossible to prove that Ulam's question has a positive answer.

5. FUNCTIONAL ANALYSIS. Ulam's problem is related to some problems in this field [**1, 12, 14, 18**]. We will focus on one of them. Consider the Banach space ℓ_∞ of all bounded sequences of real numbers, with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \sup_n |x_n|$, and its subspace c_0 consisting of all sequences in ℓ_∞ that converge to 0. Recall that the quotient space X/Y of a Banach space by its closed subspace Y is the quotient linear space endowed with the norm $\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$. By a classical result of Parovichenko in topology, under CH the quotient space ℓ_∞/c_0 serves as a universal space for all Banach spaces X of cardinality $\mathfrak{c}$.

Theorem 3. *Under the continuum hypothesis, every Banach space X such that* $|X| = \mathfrak{c}$ *is isometric to a subspace of the quotient space* ℓ_∞/c_0.

We will show, through a simplified version of what was done in [**18**] and subsequently in [**12**], that the statement above fails if Ulam's question has a negative answer. First, let us introduce some notation.

Suppose that we have a set $E \subset \mathbb{R}^2$ that does not belong to the σ-algebra generated by rectangles. We have $E = \{(a, b) \in E : a < b\} \cup \{(a, b) \in E : a = b\} \cup \{(a, b) \in E : a > b\}$. One of those three sets does not belong to the rectangle σ-algebra. It cannot be the central one since it is the graph of a function. So it has to be either the first or the last. By symmetry, we can suppose that it is the first set. So, we suppose that in fact $E \subset \{(a, b) \in \mathbb{R}^2 : a < b\}$.

Lemma 9. *There exists a Banach space* X_E *of cardinality* $\mathfrak{c}$ *and vectors* $\{e_a : a \in \mathbb{R}\}$ *of* X_E *such that for all* $a < b$*:*

1. $\|e_a + e_b\| = 2$ *if* $(a, b) \in E$*;*
2. $\|e_a + e_b\| = 1$ *if* $(a, b) \notin E$.

Proof. Given a bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define its norm by the formula

$$\|f\| = \max\left(\sup_a |f(a)|, \sup_{(a,b)\in E} |f(a) + f(b)|\right).$$

It is easy to see that if Y is the family of all bounded functions, then $(Y, \|\cdot\|)$ is a Banach space. For each $a \in \mathbb{R}$, let $e_a \in Y$ be the characteristic function of the point a; that is, $e_a(a) = 1$ and $e_a(b) = 0$ if $b \neq a$. Note that the vectors e_a satisfy the required conditions. To finish the proof, we define X_E to be the closed subspace of Y generated by all the vectors e_a. Then $|X_E| = \mathfrak{c}$ because X_E is the set of all limits of sequences of rational linear combinations of the vectors e_a (the cardinality of the space Y is bigger than $\mathfrak{c}$). ∎

Suppose that we have an isometry $T : X_E \longrightarrow \ell_\infty/c_0$. For every $a \in \mathbb{R}$, let $S(a) = (S(a)_n)_{n\in\mathbb{N}} \in \ell_\infty$ be any representative of $T(e_a)$ in the quotient space ℓ_∞/c_0. The point here is that we can calculate the norm of $T(e_a)$ in l_∞/c_0 by the formula $\limsup_n |S(a)_n|$.

In this way, we have defined a function $S : \mathbb{R} \longrightarrow \ell_\infty$. We shall consider the sets

$$A_n = \{a \in \mathbb{R} : S(a)_n > 2/3\}, \quad B_n = \{a \in \mathbb{R} : S(a)_n < -2/3\}$$

and prove that

$$E = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n \times A_n \cup \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n \times B_n, \tag{1}$$

and this will be a contradiction with our assumption that E is not in the σ-algebra generated by rectangles.

Take $a < b$ such that $(a, b) \in E$. Then $\|e_a + e_b\| = 2$ and $\|T(e_a + e_b)\| = 2$ (because T is an isometry). It follows that $\limsup_n |S(a)_n + S(b)_n| = 2$. Consider the case when $\limsup_n (S(a)_n + S(b)_n) = 2$. Then $S(a)_n > 2/3$ and $S(b)_n > 2/3$ for infinitely many n (because $|S(a)_n|, |S(b)_n| \leq 7/6$ for almost all n). Hence, $(a, b) \in A_n \times A_n$ happens infinitely often, which means that (a, b) belongs to the first set of the right-hand side of (1). The other case, when $\limsup_n (S(a)_n + S(b)_n) = -2$, follows by a symmetric argument.

To prove the reverse inclusion, take $a < b$ such that $(a, b) \in \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n \times A_n$. Then $a, b \in A_n$ for infinitely many n, so $\limsup_n |S(a)_n + S(b)_n| > 1$. We conclude that $\|e_a + e_b\| > 1$ and therefore $(a, b) \in E$.

Remark 10. With extra effort, and considering Ulam's problem in $\mathbb{R}^n$, instead of only on $\mathbb{R}^2$, one can even obtain Banach spaces of cardinality $\mathfrak{c}$ that are not *isomorphic* to subspaces of ℓ_∞/c_0 [**12, 18**]. With completely different techniques, other such examples were found in [**3**]. Some connections of Ulam's problem to the existence of universal Banach spaces were already investigated by Mauldin [**14**].

6. FINAL REMARKS.

Ulam's problem can be generalized by asking for which sets X do all subsets of $X \times X$ belong to the σ-algebra generated by rectangles. Such a property is invariant under arbitrary bijections, so it only depends on the cardinality of X. Such cardinalities are called Kunen cardinals in [**1**], following the study in [**13**]. It is not difficult to see that cardinals larger than $\mathfrak{c}$ cannot be Kunen—the reader may wish to prove that if the diagonal in $X \times X$ is in the rectangle σ-algebra, then $|X| \leq \mathfrak{c}$. The countable cardinal is obviously Kunen, so Ulam's question is really about cardinals between the countable and the continuum. It is not difficult to verify that the proof given in Section 3 shows, without using CH or any additional axiom, that there exists an uncountable set X such that all subsets of $X \times X$ belong to the σ-algebra generated by rectangles (take X the first uncountable initial segment of $\mathbb{R}$ in some well order). In set-theoretic language, that is to say that ω_1 is a Kunen cardinal. For more information, see [**1, 12, 13, 18**].

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