A dichotomy for the spaces of probability measures

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Terminology and notation

K denotes a compact Hausdorff space, $P(K)$ is the family of regular probability measures on $Bor(K)$; every $\mu \in P(K)$ satisfies $\mu(B) = \sup\{\mu(F) : F = \overline{F} \subseteq B\}$ for $B \in Bor(K)$.
$P(K) \subseteq C(K)^*$ is equipped with the *weak** topology, generated by mappings $P(K) \ni \mu \to \mu(g)$, $g \in C(K)$.

Maharam types

A measure $\mu \in P(K)$ is **of type** κ if $L_1(\mu)$ has density κ; equivalently, κ is the least cardinality of a family $\mathcal{A} \subseteq Bor(K)$ such that $\inf\{\mu(A \triangle B) : A \in \mathcal{A}\} = 0$ for every $B \in Bor(K)$.

Local character

If X is any topological space and $x \in X$ then $\chi(x, X)$ denotes the minimal size of a local base at x.

Dichotomy

Let M be a compact and convex subset of $P(K)$. For any κ of uncountable cofinality, either

- there is $\mu \in M$ such that $\chi(\mu, M) < \kappa$, or
- there is $\mu \in M$ which is of type $\geq \kappa$.

Proof. Assume $\chi(\mu, M) \geq \kappa$ for every $\mu \in M$. Construct $g_\xi \in C(K)$ and $\mu_\xi \in M$ for $\xi \leq \kappa$ so that

(i) for $\alpha < \xi \leq \kappa$ and $f \in \mathcal{F}_\alpha = \{g_\beta, |g_\beta - g_{\beta'}| : \beta, \beta' \leq \alpha\}$ we have $\mu_\xi(f) = \mu_\alpha(f)$;

(ii) for $\xi < \kappa$ we have $\inf_{\alpha<\xi} \mu_\xi(|g_\xi - g_\alpha|) > 0$.

Then the family $\{g_\xi : \xi < \kappa\}$ will witness that the measure μ_κ is of type κ. At the step ξ we consider

$$M_\xi = \bigcap_{\alpha<\xi} \{\mu \in M : \mu(f) = \mu_\alpha(f) \text{ for } f \in \mathcal{F}_\alpha\} \neq \emptyset.$$

M_ξ has more than one element. Take different $\mu, \mu' \in M_\xi$ and set $\mu_\xi = (\mu + \mu')/2$. Then $\bigcup_{\alpha<\xi} \mathcal{F}_\alpha$ is not dense in $L_1(\mu_\xi)$, and we can find g_ξ as in (ii).

Uniformly regular measures

$\chi(\mu, P(K)) = \omega$ means that μ is a uniformly regular measure, i.e. there is a countable family $\mathcal{Z}$ of closed G_δ subsets of K such that

$$\mu(U) = \sup\{\mu(Z) : Z \subseteq U, Z \in \mathcal{Z}\}$$

for every open $U \subseteq K$ (R. Pol).

Corollary for $\kappa = \omega_1$ (essentially Borodulin-Nadzieja)

Every compact space [without isolated points] K carries either a uniformly regular [nonatomic] measure or a measure of uncountable type.

Mappings onto cubes, using Fremlin's and Talagrand's results

For every compact space K,

- *assuming $MA(\omega_1)$, either $P(K)$ has points of countable character or K can be mapped onto $[0,1]^{\omega_1}$.*
- either $P(K)$ has points of character $\leq \omega_1$ or $P(K)$ can be mapped onto $[0,1]^{\omega_2}$.

Cardinal $\mathfrak{p}$

$\mathfrak{p}$ is defined so that whenever $\gamma < \mathfrak{p}$ and $(N_\xi)_{\xi<\gamma}$ is a family of subsets of $\mathbb{N}$ with $\bigcap_{\xi\in I} N_\xi$ infinite for every finite $I \subseteq \gamma$ then there is an infinite $N \subseteq \mathbb{N}$ such that $N \subseteq^* N_\xi$ for every $\xi < \gamma$.

When $\chi(x, X) < \mathfrak{p}$

If x is in the closure of a countable set $A \subseteq X$ and $\chi(x, X) < \mathfrak{p}$ then $x = \lim_n a_n$ for some $a_n \in A$.

Haydon on Grothendieck spaces

If K is an infinite compact space and $C(K)$ is Grothendieck then K carries a measure of type $\geq \mathfrak{p}$.

Haydon-Levy-Odell

For any compact space K either

- every sequence $(\mu_n)_n$ in $P(K)$ contains a *weak** converning block subsequence, or
- there is $\mu \in P(K)$ of type $\geq \mathfrak{p}$.

Proof. For a fixed $(\mu_n)_n$ in $P(K)$ consider

$$M = \bigcap_n \overline{\mathrm{conv}}\{\mu_k : k \geq n\}.$$

If $\mu \in M$ and $\chi(\mu, M) < \mathfrak{p}$ then there is a block converging subsequence. Otherwise we apply our dichotomy for $\kappa = \mathfrak{p}$.