Baire measurability in some $C(K)$ spaces

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The weak topology in Banach spaces

If $(X, ||\cdot||)$ is a Banach space then the local base at $0 \in X$ of the weak topology on X is generated by the sets (for $x^* \in X^*$, $\varepsilon > 0$),

$$V(x^*, \varepsilon) = \{x \in X : |x^*(x)| < \varepsilon\}.$$

Note that the ball $B_X = \{x \in X : ||x|| \le 1\}$ is weakly closed:

$$B_X = \bigcap_{||x^*|| \le 1} \{x \in X : x^*(x) \le 1\}.$$

Borel structures in Banach spaces

- If X is a separable Banach space then $Bor(X, weak) = Bor(X, ||\cdot||)$.
- There are nonseparable Banach spaces X for which $Bor(X, weak) = Bor(X, norm)$; this is so for every X admitting an equivalent LUR(="good") norm.

Borel structures in $C(K)$ spaces

For a compact space K we can equipp $C(K)$ with three natural topologies:
$(C(K), ||\cdot||)$; $(C(K), weak)$; $(C(K), \tau_p)$, and we have
$Bor(C(K), ||\cdot||) \supseteq Bor(C(K), weak) \supseteq Bor(C(K), \tau_p)$.

Sample result and problems on Borel structures

- **Edgar:** $Bor(C(2^\kappa), \tau_p) = Bor(C(2^\kappa), ||\cdot||)$.
- **Marciszewski & Pol:** $Bor(C(S), \tau_p) \neq Bor(C(S), weak) \neq Bor(C(S), ||\cdot||)$, where, if S is the Stone space of the measure algebra. In fact, $\{g \in C(S) : \int g \, \mathrm{d}\lambda > 0\}$ is not τ_p-Borel. Recall that $C(S) \simeq C(\beta\omega) \equiv l_\infty$.
- **Talagrand:** $Bor(C(\beta\omega), weak) \neq Bor(C(\beta\omega), ||\cdot||)$.
- **Question:** $Bor(C(\omega^*), \tau_p) \neq Bor(C(\omega^*), weak)$? (yes, under CH) $Bor(C(\beta\omega), \tau_p) \neq Borel(\beta\omega, weak)$?

Baire structures

In any Banach space X, $Baire(X, weak)$ is the least σ-algebra making all $x^* \in X^*$ measurable, i.e. $Baire(X, weak)$ is generated by the sets of the form

$$L(x^*, a) = \{x \in X : x^*(x) < a\}.$$

In particular, $Baire(C(K), weak)$ is generated by

$$L(\mu, a) = \{g \in C(K) : \int g \,\mathrm{d}\mu < a\},$$

while $Baire(C(K), \tau_p)$ is generated by

$$L(t, a) = \{g \in C(K) : g(t) < a\}.$$

Weak Baire versus weak Borel

$Baire(X, weak) = Borel(X, weak)$ whenever X is separable.
Typically, $Baire(X, weak)$ is much smaller than $Borel(X, weak)$.
Take, for instance $X = l_\infty$.

Theorem (Fremlin)

$Baire(l_1(\omega_1), weak) = Borel(l_1(\omega_1), weak).$

The space $C(2^{\omega_1})$

Theorem.

$$Baire(C(2^{\omega_1}), \tau_p) = Borel(C(2^{\omega_1}), \tau_p) = Borel(C(2^{\omega_1}), ||\cdot||),$$

so all the Baire and Borel structures coincide.

Basic Lemma. *Every closed $F \subseteq 2^{\omega_1}$ is a decreasing intersection of a sequence $(F_p)_{p\in\mathbb{N}}$ of closed separable subspaces $F_p \subseteq 2^{\omega_1}$.*

Cardinals with Kunen's rectangle property

Write $R(\kappa)$ if

$$\mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa) = \mathcal{P}(\kappa \times \kappa),$$

i.e. the family of all rectangles $\{A \times B : A, B \subseteq \kappa\}$ generates the σ-algebra of all subsets of $\kappa \times \kappa$.

- If $R(\kappa)$ then $\kappa \leq \mathfrak{c}$.
- $R(\omega_1)$.
- $R(\mathfrak{c})$ under MA.
- Consistently, $\mathfrak{c} = \omega_2$ and $\neg R(\mathfrak{c})$.

Fremlin's result and a corollary

$Baire(l_1(\kappa), weak) = Bor(l_1(\kappa), weak)$ iff $R(\kappa)$.
Since $l_1(\omega_1) \hookrightarrow C(2^{\omega_1})$, if
$Baire(C(2^\kappa), \tau_p) = Bor(C(2^\kappa), \tau_p)$ then $R(\kappa)$.

Theorem

$Baire(C(2^\kappa), \tau_p) = Bor(C(2^\kappa), \tau_p)$ if (and only if) $R(\kappa)$.

Baire measurability of the norm

If X is a Banach space then the norm $||\cdot|| : X \to \mathbb{R}$ is $Baire(X, weak)$-measurable iff $B_X = \{x \in X : ||x|| \leq 1\}$ is weakly Baire set. Since

$$B_X = \bigcap_{||x^*|| \leq 1} \{x \in X : x^*(x) \leq 1\}$$

this is so if the intersection is countable.

$$\boxed{(B_{X^*}, weak^*) \text{ sep.}} \Rightarrow \boxed{B_X \text{ weakly Baire}} \Rightarrow \boxed{(X^*, weak^*) \text{ sep.}}$$

Baire measurability of the norm in $C(K)$

Let $\mathfrak{A}$ be a Boolean algebra and $K = \mathrm{ULT}(\mathfrak{A})$ its Stone space.
Let $P(\mathfrak{A})$ be the space of finitely additive prob. measures on $\mathfrak{A}$.
$P(\mathfrak{A})$ is compact in the topology inherited from $P(\mathfrak{A}) \subseteq [0,1]^{\mathfrak{A}}$.
Every $\mu \in P(\mathfrak{A})$ defines uniquely a functional from $C(K)^*$.

- $\mu \in P(\mathfrak{A})$ is *strictly positive* if $\mu(a) > 0$ for $a \in \mathfrak{A}^+$.
- If $\mu \in P(\mathfrak{A})$ then $(a, b) \to \mu(a \triangle b)$ is a (pseudo)metric on $\mathfrak{A}$.
- Say that $\mu \in P(\mathfrak{A})$ is *of countable type* if $\mathfrak{A}$ is separable in that pseudometric.

(1)$\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), where
$K = \mathrm{ULT}(\mathfrak{A})$, $B_{C(K)} = \{g \in C(K) : ||g|| \le 1\}$, $P(\mathfrak{A}) \subseteq C(K)^*$

1. there is a strictly positive $\mu \in P(\mathfrak{A})$ of countable type;
2. $P(\mathfrak{A})$ is separable;
3. $B_{C(K)}$ is weakly Baire;
4. there is a sequence $\mu_n \in P(\mathfrak{A})$ distinguishing $g \in C(K)$.

- **Mägerl & Namioka:** $P(K)$ is separable iff there is a sequence $\mu_n \in P(\mathfrak{A})$ such that for every $a \in \mathfrak{A}^+$, $\mu_n(a) \ge 1/2$ for some n.
- **Talagrand:** (2) $\not\Rightarrow$ (1) and (4) $\not\Rightarrow$ (2) under CH.
- **Džamonja & GP:** (2) $\not\Rightarrow$ (1).
- **APR:** There is $\mathfrak{A}$ showing that (4) $\not\Rightarrow$ (2).
- Likely, the same $\mathfrak{A}$ shows (3) $\not\Rightarrow$ (2).

Construction of $\mathfrak{A}$

- Let $\mathfrak{B}$ be the measure algebra of the product measure λ on $2^{\mathfrak{c}}$.
- $|\mathfrak{B}| = \mathfrak{c}$ so we may faithfully index $\mathcal{J} = \{N_b : b \in \mathfrak{B}\}$ some independent family $\mathcal{J}$ of subsets of $\mathbb{N}$.
- Work in $\mathfrak{B}^{\mathbb{N}}$; if $a \in \mathfrak{B}^{\mathbb{N}}$ then $a = (a(n))_{n\in\mathbb{N}}$.
- Define $G_b \in \mathfrak{B}^{\mathbb{N}}$ as $G_b(n) = b$ for $n \in N_b$ and $G_b(n) = 0$ otherwise.
- $\mathfrak{A}$ is the subalgebra in $\mathfrak{B}^{\mathbb{N}}$ generated by all G_b's.
- In other words, $\mathfrak{A}$ is freely generated by G_b modulo $G_{b_1} \wedge \ldots \wedge G_{b_k} = 0$ whenever $b_1 \wedge \ldots \wedge b_k = 0$.
- $P(\mathfrak{B})$ is not separable and this implies that $P(\mathfrak{A})$ is not separable either.
- For every $n \in \mathbb{N}$, $\mu_n(a) = \lambda(a(n))$ defines $\mu_n \in P(\mathfrak{A})$.
- μ_n's distinguish functions $g \in C(K)$, where K is the Stone space of $\mathfrak{A}$.