Reflecting properties of compacta in small continuous images

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Transfinite methods in Banach spaces and algebras of operators
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joint work with
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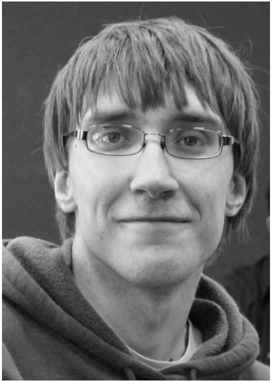
Dedication and thanks



Kamil Duszenko
Died on July 23, 2014 (aged 28).
He submitted his PhD thesis *Actions of Coxeter groups on negatively curved spaces* in 2013. There was no time for the final defence.

Kamil Duszenko Prize, **http://kamil.math.uni.wroc.pl/en/** is awarded in haematology and in mathematics (geometric group theory).
The prize was partially founded by all those who generously supported his struggle with leukaemia in 2013 and 2014.
Thank you.

Reflection problems in topology

Type 1 reflection problem

Does a topological space X has a property (P) provided all its small subspaces have property (P)?

Type 2 reflection problem, Tkachuk and Tkachenko (2012, 2015)

Does a topological space X has property (P) provided every continuous image of X of small weight has property (P)?

If $\mathfrak{A}$ is a Boolean algebra then $\mathrm{ult}(\mathfrak{A})$ is its Stone space. If K is compact zerodimensional then $\mathfrak{A} = \mathrm{clop}(K)$ is the algebra of its clopen subsets.

Type 2 problem for Boolean algebras

Suppose that every subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$, if $|\mathfrak{B}| \leq \omega_1$ then $\mathfrak{B}$ has property (P'). Does $\mathfrak{A}$ have property (P')?

Eberlein compacta, Corson compacta and ω_1

Two problems (Tkachuk and Tkachenko)

Let K be a compact space.

- Suppose that every continuous image of K of weight $\leq \omega_1$ is Eberlein compact. Is K Eberlein compact itself?
- Suppose that every continuous image of K of weight $\leq \omega_1$ is Corson compact. Is K Corson compact itself?

One (relatively consistent) answer: no

Subject to some set-theoretic assumption, there exists a compact space K which is not Corson compact but all its continuous images of weight $\leq \omega_1$ are Eberlein compacta.

K always denotes a compact Hausdorff space

Basics

1. K is **Eberlein compact** if it is homeomorphic to a weakly compact subset of some Banach space.
2. Amir-Lindenstrauss: K is Eberlein compact if for some κ it embeds into
$$c_0(\kappa) = \{x \in \mathbb{R}^\kappa : \{\alpha : |x_\alpha| \geq \varepsilon\} \text{ is finite for every } \varepsilon > 0\}.$$
3. K is **Corson compact** if there is κ such that K embeds into
$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$
4. If $n \in \omega$ then every compact subset of
$$\sigma_n(2^\kappa) = \{x \in 2^\kappa : |\{\alpha : |x_\alpha \neq 0\}| \leq n\},$$
is uniform Eberlein compact, embeds into $l_2(\kappa)$.

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The axiom

Stationary sets

$F \subseteq \gamma$ is *closed* if it is closed in the interval topology of $\gamma = \{\alpha : \alpha < \gamma\}$. F is unbounded in γ if for every $\beta < \gamma$ there is $\alpha \in F$ such that $\beta < \alpha$. $S \subseteq \gamma$ is *stationary* if $S \cap F \neq \emptyset$ for every closed and unbounded $F \subseteq \gamma$.

Axiom (*)

There is a stationary set $S \subseteq \omega_2$ such that

1. $\mathrm{cf}(\alpha) = \omega$ for every $\alpha \in S$;
2. $S \cap \beta$ is not stationary in β for every $\beta < \omega_2$ with $\mathrm{cf}(\beta) = \omega_1$.

Remarks on (*)

1. (*) follows from Jensen's principle $\square_{\omega_1}$ and hence it holds in the constructible universe.
2. (*) is *more true than untrue*, to prove the consistency of $\neg(*)$ one needs large cardinals (see Magidor 1982).

The result

Theorem

Under (*) there is a scattered compact space K with $K^{(3)} = \emptyset$ such that

1. K is not Corson compact;
2. whenever L is a continuous image of K with $w(L) \leq \omega_1$ then L is uniform Eberlein compact;
3. for every $Y \subseteq K$, if $|Y| \leq \omega_1$ then $\overline{Y}$ is uniform Eberlein compact.

Bell & Marciszewski: a scattered Eberlein compact space of height $\leq \omega + 1$ is uniform Eberlein compact.

The main idea

The construction: Fix a set $S \subseteq \omega_2$ granted by (*).

For $\alpha \in S$ pick $(p_n(\alpha))_{n<\omega}$ such that $p_n(\alpha) \nearrow \alpha$. Let $A_\alpha = \{p_n(\alpha) : n < \omega\}$, $\mathfrak{A}$ be the algebra of subsets of ω_2 generated by finite subsets together with $\{A_\alpha : \alpha \in S\}$; $K = \mathrm{ult}(\mathfrak{A})$.

1. Prove that for any family $\mathscr{G} \subseteq \mathfrak{A}$ generating $\mathfrak{A}$ there is $\xi < \omega_2$ such that $\xi \in G$ for uncountably many $G \in \mathscr{G}$.
2. Conclude that K is not Corson.
3. Prove that for every $\eta < \omega$, the sets A_α, $\alpha < \eta$ can be made disjoint by finite modifications.
4. Prove that for every $\eta < \omega$ the algebra $\mathfrak{B}_\eta \subseteq \mathfrak{A}$ generated by A_α, $\alpha < \eta$ and finite sets has a family of generators $\mathscr{G} \subseteq \mathfrak{B}$ such that every ξ is at most in two $G \in \mathscr{G}$.

New skin for the old ceremony

1. A Banach space X is WCG if $X = \overline{\mathrm{lin}}(C)$ for some weakly compact $C \subseteq X$.
2. K is Eberlein iff $C(K)$ is WCG.
3. If X is WCG then a subspace Y of X need not be WCG.
4. **Marián Fabian** (1987): If X is Asplund and WCG then every subspace of X is WCG.

Theorem

Under (*) there is a Banach space X of density ω_2 which is not WCG but all its subspaces of density $\leq \omega_1$ are WCG.

Countable functional tightness

Definition

For topological space X, $t_0(X) = \omega$ if for every $f : X \to \mathbb{R}$, if $f|A \in C(A)$ for any countable $A \subseteq X$ then $f \in C(X)$.

Lemma

If $f \in C(A)$ for every countable $A \subseteq X$ then $f|Y \in C(Y)$ for every separable $Y \subseteq X$. In particular, $t_0(X) = \omega$ for separable X.

Theorem

1. **Uspenskii** (1983): $t_0(2^\kappa) = \omega$ iff there are no measurable cardinals $\leq \kappa$.
2. **Talagrand** (1984): the Banach space $C(2^\kappa)$ is realcompact in its weak topology iff there are no measurable cardinals $\leq \kappa$.
3. **Mazur**: If there are no weakly inaccessible cardinals $\leq \kappa$ then every sequentially continuous $f : 2^\kappa \to \mathbb{R}$ is continuous
4. Cf. **Plebanek** (1991,1992).

The result

Let λ be the Lebesgue measure on $[0,1]$,
$\mathscr{N} = \{B \in Borel[0,1] : \lambda(B) = 0\}$.
$\mathrm{cov}(\mathscr{N}) > \omega_1$ means that $[0,1]$ cannot be covered by ω_1 null sets.
Let $\mathfrak{A}$ be the measures algebra.

Theorem

Let S be the Stone space of the measure algebra $\mathfrak{A}$.

1. Then $t_0(S) > \omega$.
2. Suppose that $\mathrm{cov}(\mathscr{N}) > \omega_1$. Then every continuous image L of S with $w(L) \leq \omega_1$ is separable so $t_0(L) = \omega$.

Under MA_{ω_1} this answers in the negative the following question of **Tkachuk and Tkachenko** (2015):

Problem

Let K be a compact space such that $t_0(L) = \omega$ for every its continuous image L with $w(L) \leq \omega_1$. Does this imply that $t_0(K) = \omega$?