Weakly compact sets in separable Banach spaces

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Banach spaces

X denotes a (separable) Banach space and B_X is its unit ball.
X^* is the dual space of all continuous functionals on X.

1. The space l_p of all series summable in the $p > 1$ power;
   $||x|| = (\sum_n |x(n)|^p)^{1/p}$. $(l_p)^* = l_q$, $(l_p)^{**} = l_p$.
2. The space c_0 of sequences converging to 0.
   $||x|| = \sup_n |x(n)|$, $(c_0)^* = l_1$.
3. The space l_1 of absolutely summable series.
   $||x|| = \sum_n |x(n)|$, $(l_1)^* = l_\infty$.
4. The space $C[0,1]$ of continuous functions with the sup norm.

Weakly compact sets in Banach spaces

The topology *weak* on X is the weakest topology making all $x^* \in X^*$ continuous. Sets of the form $V(x^*) = \{x \in X : |x^*(x)| < \varepsilon\}$, $x^* \in X^*$ are the subbase of the weak topology at $0 \in X$.

Notation

$\mathscr{K}(B_X)$ denotes the family of weakly compact subsets of the ball.

Main objective

Classify separable Banach spaces according to properties of $\mathscr{K}(B_X)$, considered as a set partially ordered by inclusion and/or some other relations.

Example

$B_X \in \mathscr{K}(B_X)$ iff $X^{**} = X$.

Comparing posets $(P, \leq)$ and $(Q, \leq)$: Tukey reductions

Definition

We say that P is Tukey reducible to Q and write $P \preccurlyeq Q$ if there is a function $f : P \to Q$ such that $f^{-1}(B)$ is bounded in P whenever $B \subseteq Q$ is bounded.

In other words...

$P \preccurlyeq Q$ means for every $q \in Q$ there is $h(q) \in P$ such that for every $x \in P$, if $f(x) \leq q$ then $x \leq h(q)$.
$h : Q \to P$ satisfies: $h(C)$ is cofinal in P for every cofinal $C \subseteq Q$.
Q is reacher as a cofinal structure and $\mathrm{cf}(P) \leq \mathrm{cf}(Q)$.
Here $\mathrm{cf}(Q)$ denotes the least cardinality of a set $C \subseteq Q$ which is **cofinal**, i.e. for every $q \in Q$ there is $c \in C$ with $q \leq c$.

Notation

P and Q are Tukey equivalent, $P \sim Q$, whenever $P \preccurlyeq Q$ and $Q \preccurlyeq P$.
$P \prec Q$ means $P \preccurlyeq Q$ but **not** $Q \preccurlyeq P$.

Tukey reductions, continued

Some simple posets

$\{0\} \prec \omega \prec \omega^\omega \prec \mathscr{K}(\mathbb{Q}) \prec [\mathfrak{c}]^{<\omega}$.

ω^ω

For $g_1.g_2 \in \omega^\omega$, $g_1 \leq g_2$ if $g_1(n) \leq g_2(n)$ for **every** $n \in \omega$.

For the properties of $\mathscr{K}(\mathbb{Q})$ see **Fremlin** 91 and **Gartside & Mamatelashvili** 2014.

Remarks on cofinalities

- We have $\mathrm{cf}(\omega) = \omega$, $\mathrm{cf}([\mathfrak{c}]^{<\omega}) = \mathfrak{c}$
- $\mathrm{cf}(\omega^\omega)$ is denoted by $\mathfrak{d}$.
- $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$.
- **Fremlin** 1991: $\mathrm{cf}(\mathscr{K}(\mathbb{Q})) = \mathfrak{d}$.

Classification of $\mathscr{K}(E)$ and its consequence

Theorem (Fremlin 91)

If E is coanalytic in some Polish space then either

1. $\mathscr{K}(E) \sim \mathscr{K}[0,1]$ $\sim \{0\}$ (E compact), or
2. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{N})$ $\sim \omega$ (E loc. compact noncompact),
3. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{R}\setminus\mathbb{Q})$ $\sim \omega^\omega$ (E Polish not loc. compact), or
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$ (E coanalytic but not Polish).

Corollary

If X is a Banach space with X^* separable then

1. $\mathscr{K}(B_X) \sim \{0\}$ (X reflexive), or
2. does not occur: weakly loc. compact implies compact.
3. $\mathscr{K}(B_X) \sim \omega^\omega$ (X not reflexive, has PCP), or
4. $\mathscr{K}(E) \sim \mathscr{K}(\mathbb{Q})$ (X does not have PCP).

Proof.

If X^* is separable then $(B_{X^{**}}, weak^*)$ is compact metric and $(B_X, weak)$ is $F_{\sigma\delta}$.

X has PCP if for every weakly closed bounded $A \subseteq X$, $(A, weak) \to (A, norm)$ has a point of conntinuity.

Edgar & Wheeler: $(B_X, weak)$ is Polish iff X^* is separable and X has PCP.

Possible Tukey classification of Banach spaces

Example

If $X = C[0,1]$ then $\mathscr{K}(B_X) \sim [\mathfrak{c}]^{<\omega}$.

Conjecture

If X is a separable Banach space then $\mathscr{K}(B_X)$ is Tukey equivalent to one of the following:

$$\{0\}, \quad \omega^\omega, \quad \mathscr{K}(\mathbb{Q}), \quad [\mathfrak{c}]^{<\omega}.$$

SWCG Banach space

Definition

A Banach space X is WCG if $X = \overline{\text{lin}}(K)$ for some $K \in \mathscr{K}(X)$.

Every separable X is WCG...

Definition

A Banach space X is SWCG if there is $L \in \mathscr{K}(X)$ such that for every $K \in \mathscr{K}(X)$ and $\varepsilon > 0$ there is n such that $K \subseteq n \cdot L + \varepsilon \cdot B_X$.

In other words

X is SWCG if and only if there are $L_n \in \mathscr{K}(B_X)$ such that for every $K \in \mathscr{K}(B_X)$ and $\varepsilon > 0$ we have $K \subseteq L_n + \varepsilon \cdot B_X$ for some n.

Examples

$L_1[0,1]$ is *SWCG*; try $L = \{f \in L_1[0,1] : |f| \leq 1\}$.
l_1 is SWCG; try $K_n = \{x \in B_{l_1} : x(k) = 0 \text{ for } k \geq n\}$.
c_0 is not SWCG; how many weakly compact sets we need to generate c_0 strongly?

Asymptotic structures

Definition

Say that $(P, \leq_\varepsilon \colon \varepsilon > 0)$ is an **asymptotic structure** if every $\leq_\varepsilon$ is a binary relation on P and for $\eta < \varepsilon$, $x \leq_\eta y$ implies $x \leq_\varepsilon y$.

Definition

Given asymptotic structures $(P, \leq_\varepsilon \colon \varepsilon > 0)$ and $(Q, \leq_\varepsilon \colon \varepsilon > 0)$, we say that $P \preccurlyeq Q$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$(P, \leq_\varepsilon) \preccurlyeq (Q, \leq_\delta).$$

Remarks

Given an asymptotic structure $(P, \leq_\varepsilon \colon \varepsilon > 0)$ and an ordinary poset $(Q, \leq)$,

- $P \preccurlyeq Q$ means $(P, \leq_\varepsilon) \preccurlyeq (Q, \leq)$ for every $\varepsilon > 0$;
- $Q \preccurlyeq P$ means $(Q, \leq) \preccurlyeq (P, \leq_\delta)$ for some $\delta > 0$.

Asymptotic structures of weakly compact sets

Notation

$\mathbb{AK}(B_X)$ is $\mathscr{K}(B_X)$ equipped with relations $\leq_\varepsilon$, where $K \leq_\varepsilon$ means $K \subseteq L + \varepsilon \cdot B_X$.

Examples and remarks

- X is SWCG iff $\mathbb{AK}(B_X) \preccurlyeq \omega$.
- If $X = c_0$ then $\mathbb{AK}(B_X) \sim \mathscr{K}(\mathbb{Q})$. Hence $\mathrm{cf}(\mathbb{AK}(B_X)) = \mathfrak{d}$ so c_0 is strongly generated by $\mathfrak{d}$ weakly compact sets.
- If $\mathbb{AK}(B_X) \sim P$ for some poset P then $P \preccurlyeq \mathscr{K}(B_X) \preccurlyeq P^\omega$.
- To show that $P \preccurlyeq \mathbb{AK}(B_X)$ we need to define $f : P \to \mathscr{K}(B_X)$ such that for every $L \in \mathscr{K}(B_X)$ there is $p \in P$ such that whenever $f(x) \subseteq L + \varepsilon \cdot B_X$ then $x \leq p$.

Open problem

Problem

Is it true that for every *separable* X, either $\mathbb{AK}(B_X) \preccurlyeq \omega$ or $\omega^\omega \preccurlyeq \mathbb{AK}(B_X)$?

Remarks

For every Banach space X, either $\mathscr{K}(B_X) \sim \{0\}$ or $\omega^\omega \preccurlyeq \mathscr{K}(B_X)$. Assuming $\mathfrak{d} > \omega_1$, for the *nonseparable* space $X = l_1(\omega_1)$,

- neither $\mathbb{AK}(B_X) \preccurlyeq \omega$ (because X is not SWCG),
- nor $\omega^\omega \preccurlyeq \mathbb{AK}(B_X)$ (because $\mathrm{cf}(\mathbb{AK}(B_X)) = \omega_1$).

Banach spaces not containing l_1

Theorem

If a separable space X does not contain an isomorphic copy of l_1 then $\mathbb{AK}(B_X) \sim \mathscr{K}(B_X)$ and, moreover, is Tukey equivalent to either

1. $\{0\}$ (if X is reflexive), or
2. ω^ω (if X is not reflexive, X^* is separable and X has PCP), or
3. $\mathscr{K}(\mathbb{Q})$ (if X is not reflexive, X^* is separable and X does not have PCP), or
4. $[\mathfrak{c}]^{<\omega}$ (if X^* is not separable).

The proof uses a result of **López Pérez & Soler Arias 2012** and some Ramsey type results due to **Todorčević** 2010 and others.

Under analytic determinacy

Theorem

Assuming the axiom of analytic determinacy, every separable space Banach space X satisfies one of the following

1. $\mathbb{AK}(B_X) \sim \mathscr{K}(B_X) \sim \{0\}$,
2. $\omega \preccurlyeq \mathbb{AK}(B_X) \preccurlyeq \omega^\omega$ and $\mathscr{K}(B_X) \sim \omega^\omega$,
3. $\mathbb{AK}(B_X) \sim \mathscr{K}(B_X) \sim \mathscr{K}(\mathbb{Q})$,
4. $\mathbb{AK}(B_X) \sim \mathscr{K}(B_X) \sim [\mathfrak{c}]^{<\omega}$.

Theorem (under analytic determinacy)

If $\mathscr{I}$ is an analytic ideal on ω, $\mathscr{I}^\perp = \{A \subseteq \omega : A \cap I \text{ finite for } I \in \mathscr{I}\}$ then $\mathscr{I}^\perp$ is Tukey equivalent to one of the following $\{0\}, \omega, \omega^\omega, \mathscr{K}(\mathbb{Q}), [\mathfrak{c}]^{<\omega}$.

The proof is based on results on analytic gaps due to **Todorčević** and analytic multigaps due to **Avilés and Todorčević** 2013-2014.

Subspaces

Two positive results

Let Y be a subspace of X.

- $\mathscr{K}(B_Y) \preccurlyeq \mathscr{K}(B_X)$.
  Proof. $\mathscr{K}(B_Y) \ni K \to K \in \mathscr{K}(B_X)$ is Tukey because if $K \subseteq L \in \mathscr{K}(B_X)$ then $K \subseteq L \cap Y \in \mathscr{K}(B_Y)$.
- If Y is complemented in X (i.e. $X = Y \oplus Z$ for some closed Z) then $\mathbb{AK}(B_Y) \preccurlyeq \mathbb{AK}(B_X)$.
  Proof. Let $P : X \to Y$ be a projection. If $K \in \mathscr{K}(B_Y)$, $L \in \mathscr{K}(B_X)$ and $K \subseteq L + \varepsilon \cdot B_X$ then $K \subseteq P(L) + \varepsilon \cdot ||P|| \cdot B_Y$.

Following **Mercourakis & Stamaki**

There is a subspace Y of $X = L_1[0,1]$ (which is SWCG so $\mathbb{AK}(B_X) \sim \omega$) such that $\mathbb{AK}(B_Y) \sim \omega^\omega$.

Unconditional bases

- Let $E = \langle e_n : n \in \omega \rangle$ be an unconditional basic sequence in X, i.e. there is $C > 0$ such that $||\sum_{n \in J} a_n \cdot e_n|| \leq C \cdot ||\sum_{n \in J} a_n \cdot e_n||$ for any finite sets $I \subseteq J \subseteq \omega$ and any scalars $a_n \in \mathbb{R}$.
- **Lemma.** Let $\mathscr{N}(E) = \{A \subseteq \omega : (e_n)_{n \in A} \text{ is weakly null}\}$. Then $\mathscr{N}(E) \preccurlyeq \mathbb{AK}(B_X)$.
- Let $\mathscr{A}$ be an adequate family on ω, i.e. $\mathscr{A}$ is hereditary and $A \in \mathscr{A}$ whenever all finite subsets of A are in $\mathscr{A}$.
- Following **Argyros & Mercourakis** 1993 define a norm $||\cdot||$ on c_{00} by
$$||x|| = \sup_{T \in \mathscr{A}} \sum_{n \in T} |x(n)|.$$
Let X be the completion of c_{00} with respect to such a norm.
- We have $\mathscr{N}(E) \sim \mathscr{A}^{\perp} \preccurlyeq \mathbb{AK}(B_X)$.
- Consistently, there is a Banach space X and $E \subseteq X$ such that $\mathbb{AK}(E)$ is not Tukey equivalent to any of $\{0\}, \omega, \omega^{\omega}, \mathscr{K}(\mathbb{Q}), [\mathfrak{c}]^{<\omega}$.