Baire measurability in $C(2^\kappa)$

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Terminology and notation

- For any family $\mathcal{A} \subseteq \mathcal{P}(X)$ we write $\sigma(\mathcal{A})$ for the σ-algebra generated by $\mathcal{A}$.
- If $\mathcal{F} \subseteq \mathbb{R}^X$ is a family of functions ten $\sigma(\mathcal{F})$ denotes the σ-algebra generated by $\mathcal{F}$, i.e. the least σ-algebra making all $f \in \mathcal{F}$ measurable.

Baire and Borel sets

In every completely regular topological space X there are two natural σ-algebras:

- $Bor(X)$ generated by all open sets, and
- $Baire(X)$ generated by all continuous functions $X \to \mathbb{R}$.

$Baire(X) = Bor(X)$ whenever X is a metric space, in general $Baire(X) \subseteq Bor(X)$.

Banach spaces $C(K)$

For a compact space K we can equipp $C(K)$ with three natural topologies

- $(C(K), ||\cdot||)$;
- $(C(K), weak)$;
- $(C(K), \tau_p)$.

We can discuss five σ-algebras on $C(K)$. Recall that

- $Baire(C(K), \tau_p) = \sigma(\delta_x : x \in K)$, where $\delta_x(g) = g(x)$
- $Baire(C(K), weak) = \sigma(\mu : \mu \in C(K)^*)$, where $\mu(g) = \int g \, \mathrm{d}\mu$.

Borel structures in $C(2^\kappa)$

$Bor(C(2^\kappa),\tau_p) = Bor(C(2^\kappa), weak) = Bor(C(2^\kappa))$,
for every κ because $C(2^\kappa)$ has a τ_p-Kadec renorming (Edgar).

Baire structures in $C(2^\kappa)$ for $\kappa \le \mathfrak{c}$

$Baire(C(2^\kappa),\tau_p) = Baire(C(2^\kappa), weak)$,
for $\kappa \le \mathfrak{c}$ because every probability measure μ on $2^{\mathfrak{c}}$ is a *weak**-limit $\mu = \lim_n (1/n)\sum_{i\le n}\delta_{x_i}$ for some sequence $x_i \in 2^\kappa$ (Fremlin).
For $\kappa \le \mathfrak{c}$ we have thus **the** Baire σ-algebra on $C(2^\kappa)$ and its Borel σ-algebra.

Theorem

$Baire(C(2^{\omega_1}),\tau_p) = Bor(C(2^{\omega_1}),\tau_p)$ and, consequently, all the five algebras on $C(2^{\omega_1})$ coincide.

Why $Baire(C(2^{\omega_1}), \tau_p) = Bor(C(2^{\omega_1}), \tau_p)$?

Lemma

Suppose that K is such a compact space that for every $n \in \mathbb{N}$ and every closed $F \subseteq K^n$, F is a decreasing intersection of a sequence $(F_p)_{p\in\mathbb{N}}$ of closed separable subspaces $F_p \subseteq K^n$.
Then $Baire(C(K), \tau_p) = Bor(C(K), \tau_p)$.

Lemma

Every closed $F \subseteq 2^{\omega_1}$ is a decreasing intersection of a sequence $(F_p)_{p\in\mathbb{N}}$ of closed separable subspaces $F_p \subseteq 2^{\omega_1}$.

Kunen cardinals

κ is a Kunen cardinal if $\mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa) = \mathcal{P}(\kappa \times \kappa)$, i.e. $\sigma(\{A \times B : A, B \subseteq \kappa\})$ contains all subsets of $\kappa \times \kappa$.

- If κ is Kunen then $\kappa \leq \mathfrak{c}$.
- ω_1 is a Kunen cardinal.
- $\mathfrak{c}$ is Kunen cardinal under MA + non CH, but, consistently, $\mathfrak{c} = \omega_2$ is not Kunen.
- If κ is a Kunen cardinal then there is no universal measure on $\mathcal{P}(\kappa)$.

Fremlin's result and a corollary

$Baire(l_1(\kappa), weak) = Bor(l_1(\kappa), weak)$ iff κ is a Kunen cardinal.
If $Baire(C(2^\kappa), \tau_p) = Bor(C(2^\kappa), \tau_p)$ then κ is a Kunen cardinal.

Theorem - the main result

$Baire(C(2^\kappa), \tau_p) = Bor(C(2^\kappa), \tau_p)$ iff κ is a Kunen cardinal.

Corollary

$C(2^\kappa)$ is measure-compact whenever κ is a Kunen cardinal.

A Banach space E is measure compact if for every weakly measurable $f:\Omega\to E$ there is a Bochner measurable $g:\Omega\to E$ such that $x^*g=x^*f$ μ-a.e. (for any probability space (Ω,Σ,μ)). Equivalently, for every finite measure ν on $Baire(E, weak)$ there is a separable subspace E_0 such that $\mu^*(E_0)=\mu(E)$.

Remark

Assuming the absence of weakly inaccessible cardinals, $C(2^\kappa)$ is measure-compact for any κ. (Plebanek [1991])

Corollary

Under MA + non CH, $Bor(2^{\omega_1})$ is countable generated.

If $D\subseteq 2^{\omega_1}$ is a countable dense set then $Bor(2^{\omega_1})=\sigma(\delta_x : x\in D)$.