Half-filling families of finite sets

Grzegorz Plebanek (Uniwersytet Wrocławski)

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1/2-filling families

- Throughtout X denotes an infinite space.
- $Fin(X)$ denotes the family of all finite subsets of X.
- $\mathcal{A} \subseteq Fin(X)$ is *hereditary* if $a \in \mathcal{A}$ and $a' \subseteq a$ imply $a' \in \mathcal{A}$.
- **Definition.** A hereditary family $\mathcal{A}$ *is a* 1/2*-filling family on* X *if for every* $B \in Fin(X)$ *there is* $a \in \mathcal{A}$ *such that*

$$a \subseteq b \text{ and } |a| \geq \frac{1}{2}|b|.$$

- A set $Y \subseteq X$ is $\mathcal{A}$-homogenous if $Fin(Y) \subseteq \mathcal{A}$.

Schreier family

Question. If $\mathcal{A}$ is a 1/2-filling family on an infinte space X, is there an infinite set $Y \subseteq X$ which is $\mathcal{A}$-homogenous?
Answer: No. Let $\mathcal{A} = \{a \in Fin(\mathbb{N}) : |a| \leq \min a\}$. Then $\mathcal{A}$ is 1/2-filling and has no infinite homogenous set.

Fremlin's problem

Let $\mathcal{A}$ be a 1/2-filling family on an uncountable space X.

1. Does there exist an **infinite** homogenous set $Y \subseteq X$?
2. Does there exist an **uncountable** homogenous set $Y \subseteq X$?

Basic construction: filling families from the measure

- Let $F_\xi \subseteq [0,1]$ be closed sets with $\lambda(F_\xi) \geq 1/2$ for $\xi < \omega_1$.
- $$\mathcal{A} = \{a \in Fin(\omega_1) : \bigcap_{\xi \in a} F_\xi \neq \emptyset\}.$$
- $\mathcal{A}$ is $1/2$-filling on ω_1.
- Consistently, for instance under CH, F_ξ's may be taken so that $\mathcal{F} = \{F_\xi : \xi < \omega_1\}$ contains no uncountable subfamily with nonempty intersection.
- In such a case, there is no $\mathcal{A}$-homogeneous uncountable subset of ω_1.
- There are many infinite countable $\mathcal{A}$-homogeneous $Y \subseteq \omega_1$.
- Under MA(ω_1), $\mathcal{F}$ contains an uncountable subfamily with nonempty intersection; hence there are uncountable homogeneous $Y \subseteq \omega_1$ for $\mathcal{A}$.

Strong fillings

Say that heredatiry $\mathcal{A} \subseteq Fin(X)$ is a *strong* $1/2$*-filling on* X if for every $b \in Fin(X)$ and $\varphi : b \to \mathbb{N}$ there is $a \in \mathcal{A}$ such that $a \subseteq b$ and

$$\sum_{x \in a} \varphi(x) \geq \frac{1}{2} \sum_{x \in b} \varphi(x).$$

- Strong $1/2$-fillings are $1/2$-fillings (take $\varphi \equiv 1$).
- **Džamonja & GP** Every strong $1/2$-filling comes from some measure, as in the basic construction. Hence,
- Strong $1/2$-fillings do have infinite homogeneous sets.
- **Problem.** Suppose that $\mathcal{A}$ is a $1/2$-filling on uncountable X. Is there uncountable $Y \subseteq X$ such that $\mathcal{A}_Y$ is strong $1/2$-filling on Y?

Compact families in $Fin(X)$

Let $A \subseteq Fin(X)$ be hereditary. Say that $\mathcal{A}$ is **compact** if there is no infinite $\mathcal{A}$-homogeneous set $Y \subseteq X$. If $\mathcal{A}$ is compact then the space $K(\mathcal{A}) = \mathcal{A} \subseteq P(X) \simeq 2^X$,

$$P(X) \ni E \longleftrightarrow \chi_E \in 2^X,$$

is compact.

Proof. If $E \notin K(\mathcal{A})$ then there is $b \in Fin(E)$ such that $b \notin \mathcal{A}$. Then $V = \{T \in 2^X : b \subseteq T\}$ is an open neighbourhood of E disjoint from $K(\mathcal{A})$.

Application of Fremlin's problem

Problem. Do McShane and Pettis integrals coincide for functions valued in weakly compactly generated Banach spaces?

Avilés, GP, Rodríguez: No, there is a counterexample valued in a Banach space $C(K)$ for some compact space K.

- Suppose that $\mathcal{A} \subseteq Fin([0,1])$ is a compact $1/2$-filling. Then there is a counterexample in the Banach space $C(K(\mathcal{A}))$.
- Say that $\mathcal{A} \subseteq Fin([0,1])$ is a MS-filling if for arbitrary partition $[0,1] = \bigcup_{n=1}^{\infty} Y_n$ there is $a \in \mathcal{A}$ such that

$$\lambda^* \left(\bigcup_{Y_n \cap a \neq \emptyset} Y_n \right) \geq \frac{1}{2}.$$

- There is a compact MS-filling!