A NORMAL MEASURE ON A COMPACT CONNECTED SPACE

GRZEGORZ PLEBANEK

Abstract. We present a construction of a compact connected space which supports a normal probability measure.

1. Introduction

If K is a compact Hausdorff space then we denote by $P(K)$ the set of all probability regular Borel measures on K. We write $\mathcal{Z}(K)$ for the family of all closed G_δ subsets of K. Since every compact space is normal, $Z \in \mathcal{Z}(K)$ if and only if Z is a zero set, i.e. $Z = f^{-1}(0)$ for some continuous function $f : K \to \mathbb{R}$.

A measure $\mu \in P(K)$ is *normal* if μ is order-continuous on the Banach lattice $C(K)$. Equivalently, $\mu(F) = 0$ whenever $F \subseteq K$ is a closed set with empty interior ([1], Theorem 4.6.3). A typical example of a normal measure is the natural measure defined on the Stone space of the measure algebra $\mathfrak{A}$ of the Lebesgue measure λ on $[0, 1]$. Since the algebra $\mathfrak{A}$ is complete, its Stone space is extremely disconnected.

By a result from [2] if K is a locally connected compactum then no measure $\mu \in P(K)$ can be normal, cf. [1], Proposition 4.6.20. Dales et al. posed a problem that can be stated as follows (Question 2 in [1]).

Problem 1.1. *Suppose that K is a compact and $\mu \in P(K)$ is a normal measure. Must K be disconnected?*

We show below that the answer is negative, namely we prove the following result.

Theorem 1.2. *There is a compact connected space L of weight $\mathfrak{c}$ which is the support of a normal measure.*

2. Preliminaries

Recall that $\mu \in P(K)$ is said to be *strictly positive* or *fully supported by K* if $\mu(U) > 0$ for every non-empty open set $U \subseteq K$.

Lemma 2.1. *Let K be a compact space, and suppose that μ is a strictly positive measure on K such that $\mu(Z) = 0$ for every $Z \in \mathcal{Z}(K)$ with empty interior. Then μ is a normal measure.*

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Proof. Assume that there is a closed set $F \subseteq K$ with empty interior but with $\mu(F) > 0$. Then we derive a contradiction by the following observation.

Claim. Every closed set $F \subseteq K$ with empty interior is contained in some $Z \in \mathcal{Z}(K)$ with empty interior.

Indeed, consider a maximal family $\mathcal{F}$ of continuous functions $K \to [0,1]$ such that $f|F = 0$ for $f \in \mathcal{F}$ and $f \cdot g = 0$ whenever $f, g \in \mathcal{F}$, $f \neq g$. Then $\mathcal{F}$ is necessarily countable because K, being the support of a measure, satisfies the countable chain condition. Write $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ and let $f = \sum_n 2^{-n} f_n$ and $Z = f^{-1}(0)$. Then the function f is continuous so that $Z \in \mathcal{Z}(K)$. We have $Z \supseteq F$ and the interior of Z must be empty by the maximality of $\mathcal{F}$. □

If $f : K \to L$ is a continuous map and $\mu \in P(K)$ then the measure $f[\mu] \in P(L)$ is defined by $f[\mu](B) = \mu(f^{-1}(B))$ for every Borel set $B \subseteq L$.

We shall consider inverse systems of compact spaces with measures of the form

$$\langle K_\alpha, \mu_\alpha, \pi^\alpha_\beta : \beta < \alpha < \kappa \rangle,$$

where κ is an ordinal number and for all $\gamma < \beta < \alpha < \kappa$ we have

2(i) K_α is a compact space and $\mu_\alpha \in P(K_\alpha)$;
2(ii) $\pi^\alpha_\beta : K_\alpha \to K_\beta$ is a continuous surjection;
2(iii) $\pi^\beta_\gamma \circ \pi^\alpha_\beta = \pi^\alpha_\gamma$;
2(iv) $\pi^\alpha_\beta[\mu_\alpha] = \mu_\beta$.

The following summarises basic facts on inverse systems satisfying 2(i)-(iv).

Theorem 2.2. *Let K be the limit of the system with uniquely defined continuous surjections $\pi_\alpha : K \to K_\alpha$ for $\alpha < \kappa$.*

(a) *K is a compact space and K is connected whenever all the space K_α are connected.*
(b) *There is the unique $\mu \in P(K)$ such that $\pi_\alpha[\mu] = \mu_\alpha$ for $\alpha < \kappa$.*
(c) *If every μ_α is strictly positive then μ is strictly positive.*

Engelking's *General Topology* contains the topological part of 2.2 (measure-theoretic ingredients call for a proper reference). We also use the following fact on closed G_δ sets and inverse systems of length ω_1.

Lemma 2.3. *Let K be the limit of an inverse system $\langle K_\alpha, \pi^\alpha_\beta : \beta < \alpha < \omega_1 \rangle$. Then for every $Z \in \mathcal{Z}(K)$, there are $\alpha < \omega_1$ and $Z_\alpha \in \mathcal{Z}(K_\alpha)$ with $Z = \pi_\alpha^{-1}(Z_\alpha)$.*

Proof. Sets of the form $\pi_\alpha^{-1}(V)$, where $\alpha < \kappa$ and $V \subseteq K_\alpha$ is open, give the canonical basis of K (closed under countable unions). Therefore if $Z \in \mathcal{Z}(K)$ then $Z = \bigcap_n \pi_{\alpha_n}^{-1}(V_n)$ for some $\alpha_n < \omega_1$ and some open $V_n \subseteq K_{\alpha_n}$. Taking $\alpha > \sup_n \alpha_n$ we can write $Z = \bigcap_n \pi_\alpha^{-1}(W_n)$ for some open $W_n \subseteq K_\alpha$. Let $Z_\alpha = \bigcap_n W_n$. Then Z_α is G_δ in K_α, $\pi_\alpha^{-1}(Z_\alpha) = Z$ and $Z_\alpha = \pi_\alpha(Z)$ is closed. □

3. Proof of Theorem 1.2

We first describe a basic construction which will be used repeatedly.

Lemma 3.1. *Let K be a compact connected space, and let $\mu \in P(K)$ be a strictly positive measure. If $F \subseteq K$ is a closed set with $\mu(F) > 0$, then there are a compact connected space $\widehat{K}$, a strictly positive measure $\widehat{\mu} \in P(\widehat{K})$ and a continuous surjection $f : \widehat{K} \to K$ such that $f[\widehat{\mu}] = \mu$ and $\operatorname{int}((f^{-1}(F)) \neq \emptyset$.*

Proof. Let F_0 be the support of μ restricted to F, that is

$$F_0 = F \setminus \bigcup \{U : U \text{ open and } \mu(F \cap U) = 0\}.$$

Let $\widehat{K} = \{(x,t) \in K \times [0,1] : x \in F_0 \text{ or } t = 0\}$. Then $\widehat{K}$ is clearly a compact connected space and $f(x,t) = x$ defines a continuous surjection $f : \widehat{K} \to K$. Moreover, the set $f^{-1}(F)$ contains $F_0 \times [0,1]$, a set with non-empty interior. Hence $\operatorname{int}(f^{-1}(F)) \neq \emptyset$.

We can define $\widehat{\mu} \in P(\widehat{K})$ with the required property by setting

$$\widehat{\mu}(B) = \mu(f(B \cap (K \setminus F) \times \{0\})) + \mu \otimes \lambda(F \times [0,1] \cap B),$$

for Borel sets $B \subseteq \widehat{K}$, where λ is the Lebesgue measure on $[0,1]$. □

Lemma 3.2. *Let K be a compact connected space, and let $\mu \in P(K)$ be a strictly positive measure. Then there are a compact connected space $K^\#$, a strictly positive measure $\mu^\# \in P(K^\#)$ and a continuous surjection $g : K^\# \to K$ such that $g[\mu^\#] = \mu$ and $\operatorname{int}((g^{-1}(Z)) \neq \emptyset$ for every $Z \in \mathcal{Z}(K)$ with $\mu(Z) > 0$.*

Proof. Let $\{Z_\alpha : \alpha < \kappa\}$ be an enumeration of all sets $Z \in \mathcal{Z}(K)$ of positive measure. Setting $K_0 = K$, $\mu_0 = \mu$ we define inductively an inverse system $\langle K_\alpha, \mu_\alpha, \pi^\alpha_\beta : \beta < \alpha < \kappa \rangle$ satisfying 2(i)-(iv). Assume the construction for all $\alpha < \xi$.

If ξ is the limit ordinal we use Theorem 2.2 and let K_ξ be the limit of K_α, $\alpha < \kappa$, and μ_ξ be the unique measure as in 2.3.

If $\xi = \alpha + 1$ then we define K_ξ and $\mu_\xi \in P(K_\xi)$ applying Lemma 3.1 to $K = K_\alpha$, $\mu = \mu_\alpha$, $F = (\pi^\alpha_0)^{-1}(Z_\alpha)$.

Then we can define $K^\#$ and $\mu^\#$ as the limit of $\langle K_\alpha, \mu_\alpha, \pi^\alpha_\beta : \beta \leq \alpha < \kappa \rangle$ and set $g = \pi_0 : K^\# \to K$.

Indeed, if $Z \in \mathcal{Z}(K)$ and $\mu(Z) > 0$ then $Z = Z_\alpha$ for some $\alpha < \kappa$ so the interior of the set

$$(\pi^{\alpha+1}_0)^{-1}(Z_\alpha) = (\pi^{\alpha+1}_\alpha)^{-1}((\pi^\alpha_0)^{-1}(Z_\alpha),$$

is nonempty by the basic construction of Lemma 3.1. It follows that $\operatorname{int}(g^{-1}(Z_\alpha)) \neq \emptyset$, and we are done. □

We are now ready for the proof of Theorem 1.2. Let $L_0 = [0,1]$ and $\mu_0 = \lambda$. Using Lemma 3.2 we define an inverse system $\langle L_\alpha, \mu_\alpha, \pi^\alpha_\beta : \beta \leq \alpha < \omega_1 \rangle$, where $L_{\alpha+1} = (L_\alpha)^\#$

and $\mu_{\alpha+1} = (\mu_\alpha)^\#$. Consider the limit L of this inverse system with the limit measure $\nu \in P(L)$.

We shall check that ν is a normal measure using Lemma 2.1. Take $Z \in \mathcal{Z}(L)$ with $\nu(Z) > 0$. It follows from Lemma 2.3 that $Z = \pi_\alpha^{-1}(Z_\alpha)$ for some $\alpha < \omega_1$ and $Z_\alpha \in \mathcal{Z}(L_\alpha)$. Then the set $(\pi_\alpha^{\alpha+1})^{-1}(Z_\alpha)$ has non-empty interior in $L_{\alpha+1} = (L_\alpha)^\#$ and, consequently, $\mathrm{int}(Z) \neq \emptyset$.

Note that in a compact space K of topological weight $w(K) \leq \mathfrak{c}$ there are at most $\mathfrak{c}$ many closed G_δ sets. It follows from the proof of Lemma 3.2 that $w(K^\#) \leq \mathfrak{c}$ whenever $w(K) \leq \mathfrak{c}$. Therefore $w(L_\alpha) \leq \mathfrak{c}$ for every $\alpha < \omega_1$ and $w(L) = \mathfrak{c}$. This finishes the proof of our main result.

Let us remark that using Lemma 3.1 and the construction from Kunen [3] one can prove the following variant of Theorem 1.2.

Theorem 3.3. *Assuming the continuum hypothesis, there is a perfectly normal compact connected space L supporting a normal probability measure.*

Perfect normality of L means that every closed subset of L is G_δ so in particular the space L from Theorem 3.3 is first-countable.

References

[1] H.G. Dales, F.K. Dashiell Jr., A. T.-M. Lau, D. Strauss *Banach Spaces of Continuous Functions as Dual Spaces*, preprint (2014).

[2] B. Fishel, D. Papert, *A Note on Hyperdiffuse Measures*, J. London Math. Soc. s1-39 (1), (1964), 245-254.

[3] K. Kunen, *A compact L-space under CH*, Topology Appl. 12 (1981), 283-287.

Instytut Matematyczny, Uniwersytet Wrocławski

E-mail address: grzes@math.uni.wroc.pl