On isomorphisms and embeddings of Banach spaces of continuous functions

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Preliminaries

K and L always stand for compact topological spaces.
For a given K, $C(K)$ is the Banach space of all continuous real-valued functions $f : K \to \mathbb{R}$, with the usual norm: $||g|| = \sup_{x \in K} |f(x)|$.
A linear operator $T : C(K) \to C(L)$ is an **isomorphic embedding** if there are $M, m > 0$ such that for every $g \in C(K)$

$$m \cdot ||g|| \leq ||Tg|| \leq M \cdot ||g||.$$

We write $C(K) \hookrightarrow C(L)$ if such an embedding extists.
Isomorphic embedding $T : C(K) \to C(L)$ which is onto is called an **isomorphism**; we then write $C(K) \sim C(L)$ whenever the spaces are isomorphic (as Banach spaces).

Some classical results

Let $T : C(K) \to C(L)$ be an isomorphisms of Banach spaces. Then then $K \simeq L$ under any of the following additional assumptions

- **Banach-Stone:** T preserves distance, or
- **Kaplansky:** T preserves order, or
- **Gelfand-Kolmogoroff:** T preserves multiplication.

Theorem (Miljutin)

If K is an uncountable metric space then $C(K) \sim C([0,1])$.

Theorem (Bessaga-Pełczyński)

For $\alpha < \beta < \omega_1$, $C[0,\alpha] \sim C[0,\beta]$ iff $\beta < \alpha^\omega$.

Some extensions and problems

- **Amir, Cambern:** If $T : C(K) \to C(L)$ is an isomorphism with $||T|| \cdot ||T^{-1}|| < 2$ then $K \simeq L$.
- **Jarosz (1984):** If $T : C(K) \to C(L)$ is an embedding with $||T|| \cdot ||T^{-1}|| < 2$ then K is a continuous image of some compact subspace of L.

Problem

- *Calculate the Banach-Mazur distance between $C[0,1]$ and $C(2^\omega)$.*
- *Decide for which pairs of compacta K and L, $C(K) \sim C(L)$ or $C(K) \hookrightarrow C(L)$.*
- *Which classes $\mathcal{P}$ of compacta are stable under isomorphisms, i.e. $L \in \mathcal{P}$ and $C(K) \sim C(L)$ imply $K \in \mathcal{P}$?*
- *For which spaces K there is a totally disconnected L such that $C(K) \sim C(L)$?*

Some more recent results

- **Koszmider (2004):** There is a compact connected space K such that every bounded operator $T : C(K) \to C(K)$ is of the form $T = g \cdot I + S$, where $S : C(K) \to C(K)$ is weakly compact. cf. **GP(2004)**.
- **Koszmider:** Consequently, $C(K) \not\simeq C(K) \times \mathbb{R} \simeq C(K + 1)$, and $C(K)$ is not isomorphic to $C(L)$ with L totally disconnected. .
- **Aviles-Koszmider (2011):** There is a space K which is not Radon-Nikodym compact but is a continuous image of an RN compactum; it follows that $C(K)$ is not isomorphic to $C(L)$ with L totally disconnected.
- Under CH, $C(\beta\omega \setminus \omega) \equiv l_\infty/c_0$ is isometrically universal for Banach spaces of density $\leq \mathfrak{c}$; there are models of set theory in which some 'relatively small' $C(K)$ spaces cannot be embedded into $C(\beta\omega \setminus \omega)$, even isomorphically, see **Todorcevic (2011), Koszmider & Brech (2012), Krupski & Marciszewski (2012)**.

Results on positive embeddings

An embedding $T : C(K) \to C(L)$ is **positive** if $C(K) \ni g \geq 0$ implies $Tg \geq 0$.

Theorem

Let $T : C(K) \to C(L)$ be a positive isomorphic embedding. Then there is $p \in \mathbb{N}$ and a finite valued mapping $\varphi : L \to [K]^{\leq p}$ which is onto ($\bigcup_{y \in L} \varphi(y) = K$) and upper semicontinuous.

Remark: p is the integer part of $||T|| \cdot ||T^{-1}||$.
Upper semicontinuity: $\{y : \varphi(y) \subseteq U\} \subseteq L$ is open for every open $U \subseteq K$.

Results on positive embeddings 2

Corollary

If $C(K)$ can be embedded into $C(L)$ by a positive operator then $\tau(K) \leq \tau(L)$ and if L is Frechet (or sequentially compact) then K is Frechet (sequentially compact).

Here $\tau(K)$ denotes the topological tightness. Recall that $\tau(K) \leq \omega$ means: if $x \in \overline{A}$ then $x \in \overline{A_0}$ for some countable $A_0 \subseteq A$.
K is Frechet if $x \in \overline{A}$ for any $A \subseteq K$ implies that there is a sequence $a_n \in A$ coverging to x.

Main result

Theorem

Suppose that $T : C(K) \to C(L)$ is either an isomorphism or a positive embedding.
Then there is nonempty open $U \subseteq K$ such that $\overline{U}$ is a continuous image of some compact subspace of L. In fact the family of such U forms a π-base in K.

Corollary

If $C[0,1]^\kappa \sim C(L)$ then L maps continuously onto $[0,1]^\kappa$.

Corson compacta

K is **Corson compact** if $K \hookrightarrow \Sigma(\mathbb{R}^\kappa)$ for some κ, where

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$

This is equivalent to saying that $C(K)$ contains a family $\mathcal{F}$ separating points of K and point-countable (i.e. $|\{f \in \mathcal{F} : f(x) \neq 0\}| \leq \omega$ for every $x \in K$).

Theorem (Amir-Lindenstrauss)

Every Eberlein compact space (weakly compact subset of a Banach space) embeds into $c_0(\kappa) \subseteq \Sigma(\mathbb{R}^\kappa)$ for some κ and hence is Corson compact.

Corson compacta and isomorphisms

Problem

Suppose that $C(K) \sim C(L)$, where L is Corson compact. Must K be Corson compact?

- The answer is 'yes' under $MA(\omega_1)$, because the axiom implies that if $\mathcal{F} \subseteq C(K)$ is a point countable and μ is a measure on K then $\{f \in \mathcal{F} : \int f \,\mathrm{d}\mu \neq 0\}$ is countable.
- Eberlein compact are stable under isomorphisms: if L is Eberlein and $C(K) \sim C(L)$ then K is Eberlein (since, by Amir-Lindenstrauss, L Eberlein iff $C(L)$ is WCG).

Theorem (Marciszewski & GP)

If $C(K) \hookrightarrow C(L)$ and L is Corson compact then K is Corson compact itself provided K is either scattered, linearly ordered or Rosenthal compact.

Corson compacta and isomorphisms - a provisional solution

Corollary

If $C(K) \sim C(L)$ where L is Corson compact then K has a $\pi -$ base of sets having Corson compact closures. In particular, K is itself Corson compact whenever K is homogeneous.

Some technology

If μ is a finite regular Borel measure on K then μ is a continuous functional $C(K)$: $\mu(g) = \int g \, \mathrm{d}\mu$ for $\mu \in C(K)$.
In fact, $C(K)^*$ can be identified with the space of all signed regular measures of finite variation (i.e. is of the form $\mu_1 - \mu_2$, $\mu_1, \mu_2 \geq 0$).
Let $T : C(K) \to C(L)$ be a linear operator.
Given $y \in L$, let $\delta_y \in C(L)^*$ be the Dirac measure.
We can define

$$L \ni y \to \nu_y \in C(K)^*,$$

by $\nu_y(g) = Tg(y)$ for $g \in C(K)$ ($\nu_y = T^*\delta_y$).

Basic lemma

Lemma

Let $T : C(K) \to C(L)$ *be an embedding such that for* $g \in C(K)$

$$m \cdot ||g|| \leq ||Tg|| \leq ||g||.$$

Then for every $x \in K$ *and* $m' < m$ *there is* $y \in L$ *such that* $\nu_y(\{x\}) > m'$ *(recall that* $\nu_y = T^*\delta_y$*).*

Corollary

If $C(K) \sim C(L)$ *then* $|K| = |L|$*.*

Proof.

Given $x \in K$, choose $\theta(x) \in L$ such that $\nu_{\theta(x)}(\{x\}) > m/2$. Then $\theta : K \to L$ is finite-to one. □