A survey on small measures on compact spaces and Boolean algebras

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Measures on...

...compact spaces

Given a compact space K, $P(K)$ denotes the space of all probability regular Borel measures o K. Then $P(K) \subseteq C(K)^*$ is given its *weak** topology, i.e. the weakest topology making functions $P(K) \ni \mu \to \int g \, \mathrm{d}\mu$ continuous for all $g \in C(K)$.

...Boolean algebras

For a Boolean algebra $\mathfrak{A}$, $P(\mathfrak{A})$ denotes the space of all finitely additive probability measures on $\mathfrak{A}$.
$P(\mathfrak{A})$ is a closed subset of $[0,1]^{\mathfrak{A}}$; so it is a compact Hausdorff space.

- If K is totally disconnected compactum and $\mathfrak{A} = \mathrm{clopen}(K)$ then $P(K)$ is homeomorphic to $P(\mathfrak{A})$ via $\mu \to \mu|\mathfrak{A}$.
- If $\mathfrak{A}$ is a Boolean algebra then $P(\mathfrak{A})$ is homeomorphic to $P(K)$, where K is the Stone space of $\mathfrak{A}$.

Small measures on compact spaces

A measure $\mu \in P(K)$

- has **countable type** if there is a countable family $\mathcal{F} \subseteq Bor(K)$ such that $\inf\{\mu(B \triangle F) : F \in \mathcal{F}\} = 0$, for every $B \in Bor(K)$.
- is **countably determined (CD)** if there is a countable family $\mathcal{F} \subseteq \mathrm{closed}(K)$ such that $\inf\{\mu(U \setminus F) : F \subseteq U, F \in \mathcal{F}\} = 0$, for every open $U \subseteq K$.
- is **strongly countably determined (SCD)** if there is a countable family $\mathcal{F} \subseteq \mathrm{closed}\,G_\delta(K)$ such that $\inf\{\mu(U \setminus F) : F \subseteq U, F \in \mathcal{F}\} = 0$ for every open $U \subseteq K$.

SCD $\Rightarrow$ CD $\Rightarrow$ countable type.
A measure μ has countable type iff the measure algebra of μ embeds into the measure algebra of the Lebesgue measure iff $L_1(\mu)$ is a separable Banach space.
For $x \in K$ the measure δ_x is CD. δ_x is SCD iff x is a G_δ point.
Every *CD* measure has a separable support.

Small measures on Boolean algebras

A measure $\mu \in P(\mathfrak{A})$

- has **countable type** if there is a countable algebra $\mathfrak{C} \subseteq \mathfrak{A}$ such that $\inf\{\mu(a \triangle c) : c \in \mathfrak{C}\} = 0$ for every $a \in \mathfrak{A}$.
- is **countably determined (CD)** if ...
- is **strongly countably determined (SCD)** if there is a countable algebra $\mathfrak{C} \subseteq \mathfrak{A}$ such that $\inf\{\mu(a \setminus c) : c \leq a, c \in \mathfrak{C}\} = 0$ for every open $a \in \mathfrak{A}$.

The type of $\mu \in P(\mathfrak{A})$ is uncountable iff there is $\{a_\xi : \xi < \omega_1\} \subseteq \mathfrak{A}$ such that $\inf_{\xi \neq \eta} \mu(a_\xi \triangle a_\eta) > 0$.

Measures of uncountable type

Theorem (Fremlin '97)

Assume $MA(\omega_1)$.
If $\mathfrak{A}$ is a Boolean algebra then there is $\mu \in P(\mathfrak{A})$ of uncountable type iff $\mathfrak{A}$ contains an uncountable independent family.
If K is a compact space then there is $\mu \in P(K)$ of uncountable type iff K maps continuously onto $[0,1]^{\omega_1}$.

Theorem (Kunen & van Mill '95; GP '95)

The following are equivalent

1. *every measure on a Corson compact space has countable type;*
2. 2^{ω_1} *cannot be covered by ω_1 many null sets;*
3. *every measure on a first-countable compact space has countable type.*

The class $\mathcal{CD}$ of spaces admitting only CD measures

The class $\mathcal{CD}$

1. contains scattered compacta and metric compacta;
2. Pol '82: is stable under taking closed subspaces, continuous images, countable product and the functor $K \to P(K)$;
3. Mercourakis '96: contains Radon-Nikodym compacta;
4. contains Eberlein compacta (weakly compact subsets of Banach spaces;
5. Sapounakis '80: contains compact lines;
6. Brandsma & van Mill '98: contains monotonically normal compact spaces (this follows from (2), (5) and M.E. Rudin result, that every monotonically normal compact space is a continuous image of a compact line).
7. Borodulin-Nadzieja '07: contains Stone spaces of minimally generated Boolean algebras.

Measures and Rosenthal compacta

Definition

K is Rosenthal compact if K is homeomorphic to a subset of $B_1(X)$, of Baire-1 functions on some Polish space X, equipped with the topology of pointwise convergence.

Theorem

Every measure on a Rosenthal compact space has countable type.

See Bourgain's thesis from 1974, Todorcevic '99 proof from '99 and Marciszewski & GP '12.

Problem (Roman Pol)

Is every measure on a Rosenthal compact space countably determined ?

Spaces with SCD measures

Theorem (Pol '82)

Every $\mu \in P(K)$ is SCD iff $P(K)$ is first-countable.

Theorem (GP '00)

It is relatively consistent that every measure on a first-countable compact space is SCD.

Problem (David H. Fremlin, 32 £)

Is this a consequence of $MA(\omega_1)$?

Theorem (Mikołaj Krupski &GP)

Every compact space either carries a SCD measure or carries a measure of uncountable type.

Efimov spaces and measures

Definition

A Efimov space is a compact space containing no nontrivial converging sequences and no copy of $\beta\omega$

- K contains no copy of $\beta\omega$ iff K admits no continuous surjection onto $[0,1]^{\mathfrak{c}}$.
- Hence if K contains no converging sequence and every $\mu \in P(K)$ has countable type then K is Efimov.
- Dzamonja & GP '07: Under CH there is such a space K.
- Dow & Pichardo-Mendoza '09: Under CH there is a minimally generated Boolean algebra $\mathfrak{A}$ such that its Stone space K is Efimov. It follows from Borodulin-Nadzieja '07 that every $\mu \in P(K)$ is CD (in fact every nonatomic $\mu \in P(K)$ is SCD).

The topology of $P(K)$

Definition

A topological space X has countable tightness, $\tau(X) = \omega$, if for every $A \subseteq X$ and $x \in \overline{A}$ there is a countable $I \subseteq A$ such that $x \in \overline{I}$.

Problem (GP)

Assume that $\tau(P(K)) = \omega$. Does every $\mu \in P(K)$ have countable type? Suppose that $P(K)$ is a Frechet space. Is every $\mu \in P(K)$ countably determined?

Theorem (Sobota & GP)

If $P(K \times K)$ has countable tightness then every measure on K has countable type (and so does every measure on $K \times K$).

The topology of $P(K)$

Corollary

- *Every measure on a Rosenthal compact space has countable type (using Godefroy '80: if K is Rosenthal then so are $K \times K$ and $P(K \times K)$).*
- *$P(K \times K)$ has countable tightness iff $C(K \times K)$ has property (C) of Corson (see Pol '82, Frankiewicz, GP, Ryll-Nardzewski '01).*
- *For every K, either $P(K \times K)$ has uncountable tightness or a G_δ point.*

Problem (Roman Pol)

Does countable tightness of $P(K))$ imply countable tightness of $P(K \times K)$?