Almost disjoint families and Banach spaces

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University of Wrocław

36th Summer Topology Conference, July 2022

Dedicated to *Jaś*, **Kamil Duszenko** (1986 – 23/07/2014)

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3. Write $\Psi_{\mathscr{A}}$ for $\omega \cup \mathscr{A}$ and define a topology on $\Psi_{\mathscr{A}}$ by declaring that points in ω are isolated while basic neighbourhoods of $A \in \Psi_{\mathscr{A}}$ are of the form $\{A\} \cup A \setminus I$, with $I \subseteq \omega$ finite.

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There is a lot of research done on the interplay between combinatorial properties of $\mathscr{A}$ and topology of $\Psi_{\mathscr{A}}$ (or $K_{\mathscr{A}}$), see **Hrušák** [2014].

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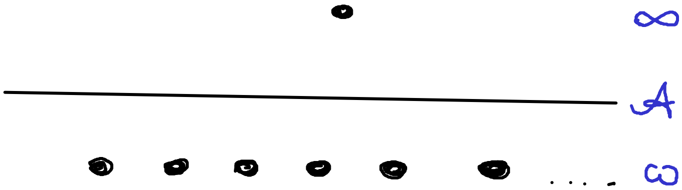

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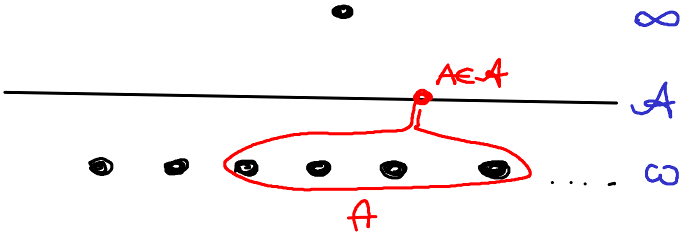

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- See also Magidor & P. [2017] for applications of almost disjoint families on ω_2 to Banach space theory.

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There are $2^{\mathfrak{c}}$ pairwise nonisomorphic Banach spaces of the form $C(K_{\mathscr{A}})$, $|\mathscr{A}| = \mathfrak{c}$.

For the second assertion: Every isomorphism $T : C(K_{\mathscr{A}}) \to C(K_{\mathscr{A}'})$ is determined by a sequence of measures μ_n on $K_{\mathscr{A}}$, where $\int g \, d\mu_n = Tg(n)$.

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2. If $C(K) \simeq C(L)$ then $\mathrm{ri}(K) \leq \mathrm{ri}(L) + 1$.
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3. Given $Z \subseteq 2^\omega$, write $\mathscr{A}(Z)$ for the AD family of branches $B(x) = \{x|n : n \in \omega\}$, where $x \in Z$.
4. The Borel complexity of Z is reflected by $\mathrm{ri}(K_{\mathscr{A}(Z)})$ so there are AD families $\mathscr{A}(Z_\xi)$ for $\xi < \omega_1$ such that $C(K_{\mathscr{A}(Z_\xi)})$ are pairwise nonisomorphic.

Only one $C(K_{\mathscr{A}})$ small space

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Theorem.

Under MA(ω_1), $C(K_{\mathscr{A}}) \simeq C(K_{\mathscr{A}'})$ whenever AD families $\mathscr{A}, \mathscr{A}'$ satisfy $|\mathscr{A}| = |\mathscr{A}'| = \omega_1$.

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Koszmider [2005] under MA, **Koszmider & Laustsen [2021]**

There is an uncountable AD family $\mathscr{A}$ such that

- Every operator $T : C(K_{\mathscr{A}}) \to C(K_{\mathscr{A}})$ is of the form $T = c \cdot I + S$, where the range of S is contained in a subspace isomorphic to c_0;
- $C(K_{\mathscr{A}}) \simeq c_0 \oplus C(K_{\mathscr{A}})$ is essentially the unique decomposition into a direct sum of infinitely dimensional summands.

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Definition.

The exact sequence above is nontrivial if $j[A]$ is not complemented in X.

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- **Correa & Tausk:** Yes, if K contains a copy of $2^{\mathfrak{c}}$.

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2. **Avilés, Marciszewski & P. [2019]:** Under CH, 'yes' for every nonmetrizable compactum K. (Consistently, Problem CCKY has a positive solution).

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- Such an exact sequence is nontrivial (c_0 is not complemented inside $C(K \cup \omega)$) iff there is no bounded extension operator $C(K) \to C(K \cup \omega)$ (in particular, there is no retraction $K \cup \omega \to L$).

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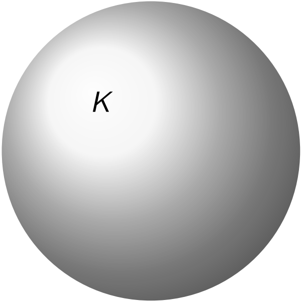

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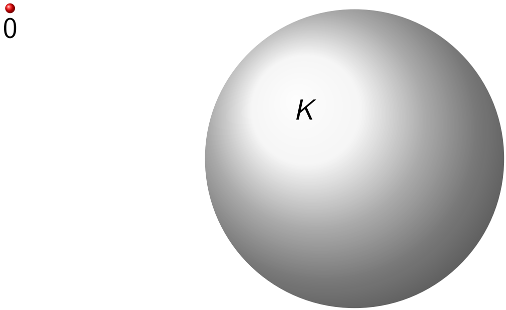

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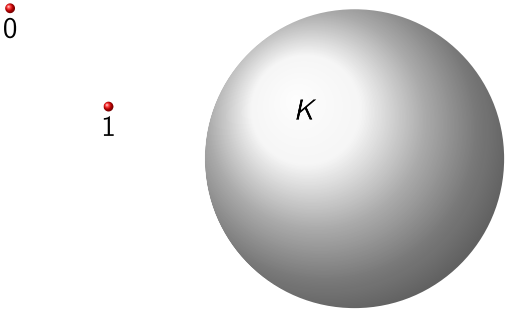

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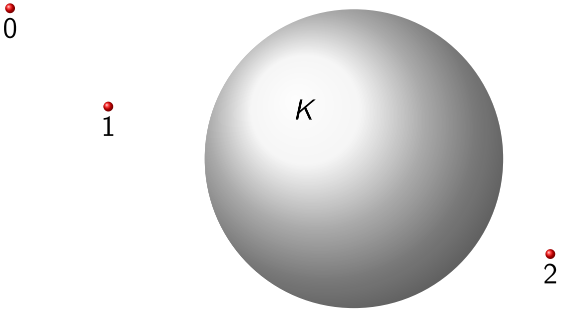

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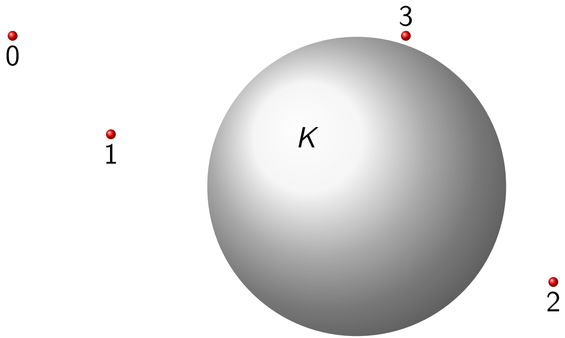

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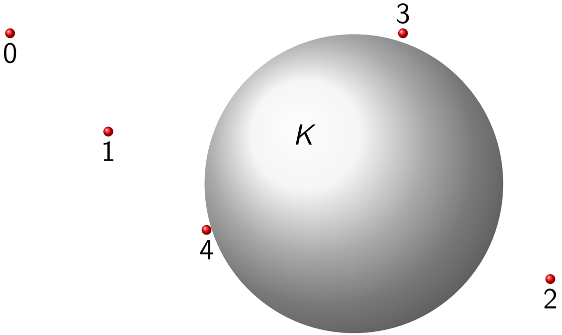

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One can find such a family $\mathscr{A}$ of cardinality $\leq \mathrm{non}(\mathscr{E})$, where $\mathscr{E}$ is the σ-ideal of subsets of $[0,1]$ generated by closed measure zero sets (see **Bartoszyński & Shelah [1992]**).

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Rosenthal [1972]

Suppose that X is a complemented subspace of $C[0,1]$ and X^* is not separable. Then $X \simeq C[0,1]$.

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- If X is a $\mathscr{C}$-space then the ball in X^* contains a closed set F such that $X \ni x \to x|F \in C(F)$ is an isomorphism.

Shape of our construction

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$\Delta = \{(n,k) : k \leq n\}$

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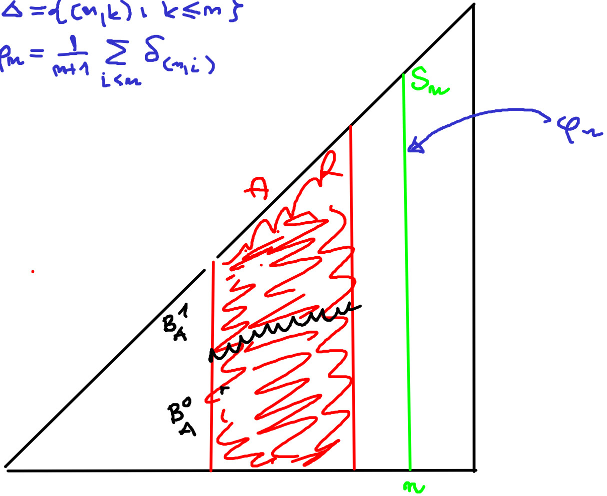

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5. Property (3) enables us to define a projection from $C(L)$ onto $\theta^\circ[C(K)]$ so $C(L) = \theta^\circ[C(K)] \oplus X$.

Remark

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Salguero-Alarcón & P. [2021]

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$$0 \longrightarrow c_0 \stackrel{j}{\longrightarrow} X \stackrel{\rho}{\longrightarrow} c_0(\mathfrak{c}) \longrightarrow 0$$

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