On isomorphisms and embeddings of $C(K)$ spaces

Grzegorz Plebanek

Insytut Matematyczny, Uniwersytet Wrocławski

Hejnice, January 2013

Preliminaries

K and L always stand for compact spaces.
For a given K, $C(K)$ is the Banach space of all continuous real-valued functions $f : K \to \mathbb{R}$, with the usual norm: $||g|| = \sup_{x \in K} |f(x)|$.
A linear operator $T : C(K) \to C(L)$ is an **isomorphic embedding** if there are $M, m > 0$ such that for every $g \in C(K)$

$$m \cdot ||g|| \leqslant ||Tg|| \leqslant M \cdot ||g||.$$

Here we can take $M = ||T||$, $m = 1/||T^{-1}||$. Isomorphic embedding $T : C(K) \to C(L)$ which is onto is called an **isomorphism**; we then write $C(K) \sim C(L)$.

Some ancient results

- **Banach-Stone:** If $C(K)$ is isometric to $C(L)$ then $K \simeq L$.
- **Amir, Cambern:** If $T : C(K) \to C(L)$ is an isomorphism with $||T|| \cdot ||T^{-1}|| < 2$ then $K \simeq L$.
- **Jarosz (1984):** If $T : C(K) \to C(L)$ is an embedding with $||T|| \cdot ||T^{-1}|| < 2$ then K is a continuous image of some compact subspace of L.
- **Miljutin:** If K is an uncountable metric space then $C(K) \sim C([0,1])$.

In particular $C(2^\omega) \sim C[0,1]$; $C[0,1] \times \mathbb{R} = C([0,1] \cup \{2\}) \sim C[0,1]$.

Some ancient problems

Problem

For which spaces K, $C(K) \sim C(K+1)$?

Here $C(K+1) = C(K) \times \mathbb{R}$.
This is so if K contains a nontrivial converging sequence:
$C(K) = c_0 \oplus X \sim c_0 \oplus X \oplus \mathbb{R} \sim C(K+1)$.
Note that $C(\beta\omega) \sim C(\beta\omega + 1)$ (because $C(\beta\omega) = l_\infty$) though $\beta\omega$ has no converging sequences.

Problem

For which spaces K there is a totally disconnected L such that $C(K) \sim C(L)$?

Some more recent results

- **Koszmider (2004):** There is a compact connected space K such that every bounded operator $T : C(K) \to C(K)$ is of the form $T = g \cdot I + S$, where $S : C(K) \to C(K)$ is weakly compact. cf. **GP(2004)**.
  Consequently, $C(K) \not\sim C(K+1)$, and $C(K)$ is not isomorphic to $C(L)$ with L totally disconnected; .
- **Aviles-Koszmider (2011):** There is a space K which is not Radon-Nikodym compact but is a continuous image of an RN compactum; it follows that $C(K)$ is not isomorphic to $C(L)$ with L totally disconnected.

Some questions

- Suppose that $C(K)$ and $C(L)$ are isomorphic. How K is topologically related to L?
- Suppose that $C(K)$ can be embedded into $C(L)$, where L has some property $\mathcal{P}$. Does K has property $\mathcal{P}$?

Results on positive embeddings

An embedding $T : C(K) \to C(L)$ is **positive** if $C(K) \ni g \geqslant 0$ implies $Tg \geqslant 0$.

Theorem

Let $T : C(K) \to C(L)$ be a positive isomorphic embedding. Then there is $p \in \mathbb{N}$ and a finite valued mapping $\varphi : L \to [K]^{\leqslant p}$ which is onto ($\bigcup_{y \in L} \varphi(y) = K$) and upper semicontinuous (i.e. $\{y : \varphi(y) \subseteq U\} \subseteq L$ is open for every open $U \subseteq K$).

Corollary

If $C(K)$ can be embedded into $C(L)$ by a positive operator then $\tau(K) \leqslant \tau(L)$ and if L is Frechet (or sequentially compact) then K is Frechet (sequentially compact).

Remark: p is the integer part of $||T|| \cdot ||T^{-1}||$.

A result on isomorphisms

Theorem

If $C(K) \sim C(L)$ then there is nonempty open $U \subseteq K$ such that $\overline{U}$ is a continuous image of some compact subspace of L. In fact the family of such U forms a π-base in K.

Corollary

If $C[0,1]^\kappa \sim C(L)$ then L maps continuously onto $[0,1]^\kappa$.

Corson compacta

K is **Corson compact** if $K \hookrightarrow \Sigma(\mathbb{R}^\kappa)$ for some κ, where

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leqslant \omega\}.$$

This is equivalent to saying that $C(K)$ contains a point-countable family separating points of K.

Problem

Suppose that $C(K) \sim C(L)$, where L is Corson compact. Must K be Corson compact?

The answer is 'yes' under $MA(\omega_1)$.

Theorem

If $C(K) \sim C(L)$ where L is Corson compact then K has a $\pi-$base of sets having Corson compact closures. In particular, K is itself Corson compact whenever K is homogeneous.

Basic technique

If μ is a finite regular Borel measure on K then μ is a continuous functional $C(K)$: $\mu(g) = \int g \, \mathrm{d}\mu$ for $\mu \in C(K)$.
In fact, $C(K)^*$ can be identified with the space of all signed regular measures of finite variation (i.e. is of the form $\mu_1 - \mu_2$, $\mu_1, \mu_2 \geqslant 0$).
Let $T : C(K) \to C(L)$ be a linear operator.Given $y \in L$, let $\delta_y \in C(L)^*$ be the Dirac measure.
We can define $\nu_y \in C(K)^*$ by $\nu_y(g) = Tg(y)$ for $g \in C(K)(\nu_y = T^*\delta_y)$.

Lemma

Let $T : C(K) \to C(L)$ be an embedding such that for $g \in C(K)$

$$m \cdot ||g|| \leqslant ||Tg|| \leqslant ||g||.$$

Then for every $x \in K$ and $m' < m$ there is $y \in L$ such that $\nu_y(\{x\}) > m'$.

An application

Theorem (W. Marciszewski, GP (2000))

Suppose that $C(K)$ embeds into $C(L)$, where L is Corson compact. Then K is Corson compact provided

- *K is linearly ordered compactum, or*
- *K is Rosenthal compact.*

Problem

Can one embed $C(2^{\omega_1})$ into $C(L)$, L Corson?

No, under MA+ non CH.
No, under CH (in fact whenever $2^{\omega_1} > \mathfrak{c}$).