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Lesson 1

FOURTH POWERS

Exercise 1.

Find all non-trivial solutions to

$$x^4+y^4=z^4+t^4$$

in positive integers less than $N=1000$. To do so, generate all (actually not all, but only those with $x \geqslant y$, but I will not write such details in future) the triples

$$(x^4+y^4,x,y),$$

sort them according to x^4+y^4 and find the repeated entries. Print only solutions with no common divisor greater than 1, i.e. so called *primitive solutions*.

Do the same for the largest N that memory of your computer allows. Use hard drive if you are really desperate to go higher than that.

Exercise 2.

Take a small prime p of the form $4k+3$ (so that -1 is not a square mod p). For good start take $p=103$ and $N=10000$.

Precompute the following tables:

Table 1 of $x^4 (\mathrm{mod}\ p)$ for $x<N$.

Put all $x<N$ in a table 2 with p rows. r-th row should contain all numbers x with $x^4 \equiv r (\mathrm{mod}\ p)$. Try doing this in 2 ways (to compare program efficiency and memory uasage): using a table with fixed size rows (this can be done in 2 ways as well: either you declare maximum size of rows or you calculate size of each row in the first run, prepare pointers to the begining of each row and store the numbers in the second run) or using linked lists.

You may also want to precompute a table of x^4 in 64-bit integers.

Now go through all r, where $0 \leqslant r < p$. For each r generate all the triples

$$(x^4+y^4,x,y)$$

with

$$x^4+y^4 \equiv r (\mathrm{mod}\ p),$$

sort them according to x^4+y^4 and find the repeated entries.

To generate such triples run throuth all $x<N$ and pick y's from the appropriate row of the table 2. Table 1 should become handy to get $x^4 (\mathrm{mod}\ p)$ quickly.

Now, play with p and N, and see how high can you go with N on your computer in 1 minute. What is the largest N that allows you to feel comfortable with using 64-bit integers?

Note, that that an overflow in 64-bit arithmetic results in computations mod 2^{64}. Therefore you can go with numbers used up to $p \cdot 2^{64}$.

Assume that each residue r takes about the same time and estimate how high could you go with appropriate p and N in one month on 10 computers like the one you are using now.

Exercise 3.

Modify your best version of the program from the previous exercise to get rid of sorting and using large numbers.

Take prime q of the form $4k+3$. For $p=103$ and $N=10000$ start with q of the order of 500000 and try to adjust it later to get the best efficiency.

Precompute addtional table of $x^4 (\mathrm{mod}\ q)$.

For each residue r mod p generate pairs (x,y) with

$$x^4+y^4 \equiv r (\mathrm{mod}\ p)$$

and store them on q linked lists according to $x^4+y^4 (\mathrm{mod}\ q)$.

When you store a pair (x,y), couple it with each pair (z,t) already on the link list to get solution (x,y,z,t) of

$$x^4+y^4 \equiv z^4+t^4 (\mathrm{mod}\ p\cdot q)\,.$$

Verify such solution by checking whether

$$x^4+y^4 \equiv z^4+t^4 (\mathrm{mod}\ v_1)\,,$$

where v_1 is another prime. Of course, you did precompute a table of $x^4 (\mathrm{mod}\ v_1)$, didn't you?

If $2\cdot N^4 > p\cdot q\cdot v_1$ use another prime v_2.

Note that you can stay in 32-bit integers except for precomputing residues of 4th powers.

Do I have to tell you to play with p, q and N and to compare this program with the previous one?

Exercise 4.

Do some minor improvements to the program.

Make sure numbers in rows in table 2 from exercise 2 are in increasing order. Since you are taking y's from there and you require $y \leqslant x$, once you find too big y there is no point in going through the rest of the row.

Any common prime divisor of x and y that is not a divisor of z and t must be of the form $8k+1$. Therefore you can skip pairs (x,y) with both numbers divisible by 2, 3, 5, and see for yourself how far can you go to gain more than loose.

Half of the rows in table 2 are empty. Can you use this fact to make any improvement? I do not know how to make a really good use of that.

Also you may consider splitting the problem into 2 cases:

1) both x and y odd,

2) x odd and y even.

Read the initial residue r from a file and after each residue is computed, update the file. That way you can kill the program at any time and continue with almost no loss.

Set up a counter to count pairs (x,y) which you are putting on linked lists. As I am sure you are already measuring time somehow, report number of pairs computed per second to get a way of measuring program efficiency.

Exercise 5.

Take $M < N$ and restrict the program to pairs (x,y) with

$$M^4 \leqslant x^4 + y^4 \leqslant N^4 .$$

That saves memory and allows to reduce p which usually speeds up the program. However M and N cannot be too close as this in turn slows the program down.

Assume you can work for one month on 10 computers like the one you are using now.

How high could you go with N?

Exercise 6.

Run a program searching for primitive solutions of the equation

$$x^4 + y^4 = z^4 + t^4 ,$$

where $x \geqslant y$, $z \geqslant t$ and $x > z$ and sort the solutions according to x (then y if necessary).

What is the 100th solution?

How many solutions are there below 100000?

Programming Project 7.

Get in touch with other people (or get hold on many computers) and split the search area to find the first 1000 solutions (I was able to find **623** solutions with

$$x^4 + y^4 < 1500000^4$$

and would be interested to get a longer list). Abandon the idea if it would be unreasonably hard.

Programming Project 8.

Analize the list of solutions to select interesting examples:

Solutions in prime numbers.

Solutions, where terms on one side have common divisor greater than 1.

Triples of equal differences of 4th powers (be careful, think how to do this before you proceed).

Before you publish the list of examples make sure there are no 3 equal sums of 2 fourth powers. You do not want to miss a historic discovery, do you?

Programming Project 9.

Modify your program to target primitive solutions with both z and t divisible by 17.

To do this, store on linked lists only pairs (x,y) with

$$x^4 + y^4 \equiv 0 (\bmod\ 17^4)$$

and x, y not divisible by 17. You may ignore pairs, where

$$\frac{x^4 + y^4}{17^4} (\bmod\ 17)$$

is one of the numbers 6, 7, 10, 11. Do not couple them with pairs already on linked list!

Then run through pairs (z,t) of numbers divisible by 17 and couple them with (x,y)'s. Do not store (z,t) on linked lists.

Conduct your search as high as you consider it reasonable.

You can do the same with 41 or any other prime of the form $8k+1$.

Let me know if you can find a primitive solution with x, y divisible by 17 and z, t divisible by 41 or any primitive solution with terms on each side having a common divisor greater than 1.