Theories with NIP, List 1

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Let T be the theory of equality with an infinite model. Consider $\varphi(x; y_1, y_2, y_3) := (x = y_1 \vee x = y_2 \vee x = y_3)$. Prove that $\text{VC-dim}(\varphi) \neq \text{VC-dim}(\varphi^{\text{opp}})$.

Problem 2. Let $\varphi(x; y)$ be a NIP formula. Prove that the set $\{\text{alt}(\varphi(x; b), I) : |b| = |y| \text{ and } I \text{ is an indiscernible sequence}\}$ is bounded by some natural number.
Comment. Thus, $\text{alt}(\varphi) := \max\{\text{alt}(\varphi(x; b), I) : |b| = |y|$ *and* I *is an indiscernible sequence*$\}$ *makes sense.*

Problem 3. Let $T = \text{Th}((M, \leq))$, where $(M, \leq)$ is an arbitrary linear order.
(i) Let $a = (a_i), b = (b_j), c = (c_k)$ be finite tuples, where c is increasing. Assume there are no i, j such that a_i and b_j are in the same open interval from the list: $(-\infty, c_0), (c_0, c_1), \dots$. Show that $\text{tp}(a/c) \cup \text{tp}(b/c) \vdash \text{tp}(ab/c)$.
(ii) Prove that T has NIP.

Problem 4. Let G be a group $\emptyset$-definable in $\mathfrak{C}$. Define G^0 as the smallest type-definable, bounded index subgroup of G which is an intersection of definable subgroups of finite index, if it exists; otherwise we say that G^0 does not exist.
(i) Show that $[G : G^0_A] \leq 2^{|T|+|A|}$ for any (small) $A \subset \mathfrak{C}$.
(ii) Show that if G^0 exists, then $G^0 = \bigcap\{G^0_A : A \subset \mathfrak{C}\}$.
(iii) Prove that G^0 exists iff $\bigcap\{G^0_A : A \subset \mathfrak{C}\}$ has bounded index.
(iv) Prove that G^0 exists iff for every $A \subset \mathfrak{C}$, $G^0_A = G^0_\emptyset$. Hence, if G^0 exists, then $G^0 = G^0_\emptyset$.

Problem 5. Assume T has NIP, and let G be a $\emptyset$-definable group. Prove that G^0 exists.

Problem 6. Prove that each global type finitely satisfiable in A is invariant over A.

Problem 7. Let $\mathfrak{C}^* \succ \mathfrak{C}$ be a bigger monster model and $p \in S(\mathfrak{C})$ be a type invariant over A.
(i) Prove that $p|\mathfrak{C}^*$ is a unique extension of p to an A-invariant type in $S(\mathfrak{C}^*)$.
(ii) Prove that $p|\mathfrak{C}^*$ does not depend on the choice of A over which p is invariant.

Problem 8. Prove that if a global, A-invariant type p is finitely satisfiable in some small set, then it is finitely satisfiable in any model $M \prec \mathfrak{C}$ containing A.

Problem 9. Prove that if $p, q \in S(\mathfrak{C})$ are A-invariant, then $p \otimes q$ is also A-invariant.

Problem 10. Prove that $\otimes$ is associative.