Theories with NIP. List 12.

We work in a monster model $\mathfrak{C}$ of a complete NIP theory T.

Problem 1. Prove that each global M-invariant Keisler measure is Borel definable over M.
Hint. Use Proposition 1 from p. 66 and Proposition 4 from p. 62 (from my notes).

Problem 2. Here, for a type $p \in S(\mathfrak{C})$ by μ_p we will mean p treated as Keisler measure. Let $p \in S_x(\mathfrak{C})$ be M-invariant and $q \in S_y(\mathfrak{C})$. Prove that $\mu_{p\otimes q} = \mu_p \otimes \mu_q$.

Problem 3. Let $\mu_x \in M_x(\mathfrak{C})$ and $\nu_y \in S_y(\mathfrak{C})$. Prove that:
(i) if μ_x and ν_y are both invariant over M, then so is $\mu_x \otimes \nu_y$;
(ii) if μ_x and μ_y are both finitely satisfiable in M, then so is $\mu_x \otimes \mu_y$;
(iii) if μ_x and μ_y are both definable in M, then so is $\mu_x \otimes \mu_y$.

Problem 4. Let $\mu_x \in M_x(\mathfrak{C})$ be M-invariant and $\nu_y \in M_x(\mathfrak{C})$. Prove that for every Borel set $B(x, y)$ over M (i.e. $B(x, y)$ is a Borel subset of $S_{xy}(M)$ or the associated set of realizations in $\mathfrak{C}^{|xy|}$ or the associated Borel subset of $S_{xy}(\mathfrak{C})$ contained in the set of types invariant over M), we have $\mu_x \otimes \nu_y(B(x, y)) = \int_{q\in S_y(M)} f(q)d\nu_y|_M$, where $f(q) := \mu_x(B(x, c))$ for some/every $c \in q(\mathfrak{C})$ (assuming that f is Borel measurable).

Problem 5. Let $\pi\colon S_{x,y}(\mathfrak{C}) \to S_x(\mathfrak{C}) \times S_y(\mathfrak{C})$ be the obvious map, and μ_x and ν_y be global Keisler measures such that μ_x is invariant over M. Let $B \subseteq B(S_x(\mathfrak{C})) \otimes B(S_y(\mathfrak{C}))$ (the product σ-alegbra of the σ-algebras of Borel sets). Prove that $\mu_x \times \nu_y(B) = \mu_x \otimes \nu_y(\pi^{-1}[B])$ (where $\mu_x \times \nu_y$ is the product measure).

Problem 6. Assume $\mu_x \in M_x(\mathfrak{C})$ is invariant over M.
(i) Prove that if μ_x is definable (over some small set), then it is definable over M.
(ii) Prove that if μ_x is finitely staisfiable (in some small model), then it is finitely satisfiable in M.

Problem 7. Let $\mathcal{F}$ be a family of measurable functions from a probability space (X, Ω, μ) to $[0, 1]$. Prove that $\mathcal{F}$ has an essential supremum which is the usual supremum of some countable subfamily of $\mathcal{F}$.

Problem 8. Deduce VC-theorem* from VC-theorem and Problem 7.

Problem 9. Prove Proposition 4 from page 62 using VC-theorem*.