Theories with NIP. List 14.

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Let E be a bounded, invariant (over $\emptyset$) equivalence relation. Show that the quotient map $\pi_E\colon \mathfrak{C} \to \mathfrak{C}/E$ factors through the map $\rho\colon \mathfrak{C} \to S(M)$ given by $\rho(a) := \mathrm{tp}(a/M)$ via a continuous map $h\colon S(M) \to \mathfrak{C}/E$ (i.e. $\pi_E = h \circ \rho$).

Problem 2. Let G by a $\emptyset$-type-definable group and $H \lhd G$ be an invariant subgroup of bounded index. Prove that G/H is a topological group (i.e. the group operation and inversion are continuous).
Hint. First, prove it for H type-definable, and then use it in general.

Problem 3. Prove that (where $\cong$ denotes *topological* isomorphisms):
(i) for any type-definable group G, G/G^0_A is always profinite;
(ii) for $M := (\mathbb{Z}, +)$ and $G(M) := \mathbb{Z}$, $G/G^{00} = G/G^0 \cong \hat{Z}$;
(iii) for $M := (\mathbb{R}, +, \cdot)$ and $G(M) := S^1$, $G/G^{00} \cong S^1$ and G/G^0 is trivial;
(iv) more generally, in the context of the fact on p. 74, $G^*/G^{*00}_G \cong G$.
Comment. The fact on p. 74 says that they are abstractly isomorphic which is the content of Problem 9 from list 13. Here, one has to show that the isomorphism is topological

Problem 4. Let $M \prec \mathfrak{C}$ be small. Assume that μ is a left invariant Keisler measure on $G(M)$ (i.e. a $G(M)$-invariant, finitely additive probability measure on $\mathrm{Def}(G)$). Prove that if μ has a unique extension to a left invariant Keisler measure on $G = G(\mathfrak{C})$, then for every $N \succ M$ the measure μ has a unique extension to a left invariant Keisler measure on $G(N)$. (In particular, the uniqueness of the extension does no depend on the choice of the monster model $\mathfrak{C}$.)

Problem 5. Prove Beth's theorem for types.
Comment. For the precise statement see Fact 2.12 in:
J. Gismatullin, K. Krupiński, On model-theoretic connected components in some group extensions, *Journal of Mathematical Logic (15), 1550009 (51 pages), 2015.*

Problem 6. Let G be a group definable in a structure M. Let $N = (M, X, \cdot)$ be M expanded by the "affine copy" X of G. Prove that:
(i) the definable subsets of M^n computed in the structure M coincide with the definable subsets of M^n computed in N (equivalently, the restriction map from $S_{M^n}(N)$ computed in the language of N to $S_n(M)$ computed in the language of M is a homeomorphism);
(ii) $S_G(M) \approx S_X(N)$ (the map is given in the proof of Lemma 6 on p. 84; here check the details);
(iii) a type $q \in S_1(M)$ does not fork over $A \subseteq M$ if and only if q treated (by (i)) as a type in $S_M(N)$ does not fork over A if and only if it does not fork over $A\beta$ for every/some $\beta \in X$.

Problem 7. Let D be $\emptyset$-definable and $p \in S_D(\mathfrak{C})$. Let $f\colon D \to D'$ be an A-definable bijection. Then we get a well-defined $f(p) \in S_{D'}(\mathfrak{C})$.

(i) Prove that p does not fork over A if and only if $f(p)$ does not fork over A.
(ii) Choose any $b \in \mathfrak{C}$. Prove that p does not fork over A if and only if it does not fork over Ab' for every $b' \models \mathrm{tp}(b/A)$.