Theories with NIP. List 4.

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Consider condintions:

(1) $\varphi(x,y)$ has IP;

(2) for some $n \in \mathbb{N} \setminus \{0\}$ and some $\eta \in 2^n$ the formula $\bigwedge_{i=0}^{n-1} \varphi(x,y_i)^{\eta(i)}$ has SOP.

Prove that $(1) \vee (2)$ implies that $\varphi(x,y)$ is unstable.

Problem 2. (i) Let $M := (\mathbb{R}, \leq) \models DLO =: T$. Show that all types in $S_n(\mathbb{R})$ are definable (equivalently, $\mathbb{R}$ is weakly stably embedded).
(ii) Prove that, in fact, $\mathbb{R}$ is stably embedded.

Problem 3. Let D be a 0-definable set. Prove that D is stably embedded iff $D_{\mathrm{ind}(\emptyset)}$ has the same definable sets as $D_{\mathrm{ind}(B)}$ for any B.

Problem 4. (i) Let $M \prec \mathfrak{C}$ and $\widehat{M} \succ M^{Sh}$ with L-reduct $\widehat{M}\restriction_L \prec \mathfrak{C}$. Prove that, up to a renaming of the language, $\widehat{M} = (\widehat{M}\restriction_L)_{\mathrm{ind}(B)}$ for some small $B \subseteq \mathfrak{C}$.
(ii) Give an example showing that $\widehat{M}$ may be a proper reduct of $(\widehat{M}\restriction_L)^{Sh}$.

Problem 5. (i) Show that if $M \prec N$, where N is $|M|^+$-saturated, then for every k the family $\{\varphi(M,b) : \varphi(x_1,\dots,x_k,y) \in L, b \subseteq N, |b| = |y|\}$ is exactly the collection of all externally definable subsets of M.
(ii) Let $N \succ M$ be $|M|^+$-saturated. Prove that $S_k^{qf}(M^{Sh}) \approx S_{ext,k}(M) \approx S_{M,k}(N)$

Problem 6. Let $A \subseteq M \prec \mathfrak{C}$, where M is $|A|^+$-saturated. Prove that A is stably embedded iff for every $(M',A') \succ (M,A)$ and every tuple m from M' the type $\mathrm{tp}_L(m/A')$ is definable.

Problem 7. Let $I \subseteq M$ be an indiscernible sequence, and let $(M',I') \succ (M,I)$. Prove that there is an ordering on I' making it into an indiscernible sequence.

Problem 8. Let $\varphi(x,y) \in L$, $A \subseteq M \prec \mathfrak{C}$, and $b \subseteq M$. Prove that $\varphi(x,b)$ has an honest definition over A (computed using M) iff there exists $\psi(x,z) \in L$ such that for every finite $A_0 \subseteq \varphi(A,b)$ there exists d in A with $A_0 \subseteq \psi(A,d) \subseteq \varphi(A,b)$.
Comment. This shows that the definition of honest definition of $\varphi(x,b)$ over A does not depend on the choice of the model M containing A and b.

Problem 9. Assume NIP. Let $D \subseteq \mathfrak{C}$ and $\varphi(x,y) \in L$ be such that $\varphi(\mathfrak{C}) \cap D$ is a linear order $\leq$. Assume that $D_{\mathrm{ind}(\emptyset)}$ is o-minimal with respect to this order. Prove that every externally definable subset of D is a union finitely many $\leq$-convex subsets.