Theories with NIP. List 5.

We work in a monster model $\mathfrak{C}$ of a complete theory T. Recall that our convention is that $\varphi(x,a) \in L(A)$ means that $\varphi(x,y) \in L$ and a is a tuple from A.

Problem 1. Let $\varphi(x,y,z) \in L(\mathfrak{C})$. Show that $\mathrm{alt}(R_{\varphi(x,y,c)}(x,y))$ computed in $\mathrm{Th}(M^{Sh})$ is less than or equal to $\mathrm{alt}(\varphi(x,y,c))$ computed in T.

Problem 2. Show that if M is the random graph, then M^{Sh} does not have quantifier elimination.

Problem 3. Let $M \prec \mathfrak{C}$ and $\varphi(x,y,b) \in L(\mathfrak{C})$. Assume that $\varphi(M,b)$ is the graph of a function. Prove that there is $\psi(x,y,d) \in L(\mathfrak{C})$ such that $\psi(M,b) = \varphi(M,b)$ and $\mathfrak{C} \models (\forall x)(\exists^{\leq 1} y)\psi(x,y,d)$.

Problem 4. Prove that T has NIP if and only if for every finite tuple b and indiscernible sequence I of cofinality at least $|T|^+$, some finite segment of I is indiscernible over b.

Problem 5. Let $\mathcal{I}$ be a linear order, and let $\mathcal{J}$ be its completion. Let $\sim$ be a convex equivalence relation on $\mathcal{I}$.
(i) Assume that $\sim$ is finite. Prove that there is a finite tuple $\bar{c} \subseteq \mathcal{J}$ such that $\sim_{\bar{c}}\restriction_{\mathcal{I}} \subseteq \sim$ and $(\forall i,j \in \mathcal{I} \setminus \bar{c})(i \sim j \iff i \sim_{\bar{c}} j)$.
(ii) Assume that $\sim$ is essentially of size κ. Prove that there is a tuple $\bar{c} \subseteq \mathcal{J}$ of length at most κ satisfying the same conditions as in (i).

Problem 6. Assume T has NIP. Let $I = (a_i)_{i \in \mathcal{I}}$ be an indiscernible sequence, and $\varphi(x_1,\dots,x_n,b) \in L(\mathfrak{C})$. Prove that there exists a coarsest finite convex equivalence relation on $\mathcal{I}$ such that for every $\bar{i},\bar{j} \in \mathcal{I}^n$ we have

$$\bar{i} \sim \bar{j} \Rightarrow\ \models (\varphi(a_{\bar{i}},b) \leftrightarrow \varphi(a_{\bar{j}},b)).$$

Comment. From the lecture we know that there is a finite convex equivalence relation on $\mathcal{I}$ satisfying the above equivalence. In this problem, the only thing to do is to deduce that there exists a coarsest such relation.

Problem 7. Assume T has NIP. Let $I = (a_i)_{i \in \mathcal{I}}$ be an indiscernible sequence, where $\mathcal{I}$ is a saturated model of DLO. Let $\mathrm{Aut}(I)$ be the group of elementary permutations of I. For every n, $\mathrm{Aut}(I)$ acts naturally on $S_n(I)$. Prove that the number of orbits under this action is at most κ.