Theories with NIP. List 7.

We work in a monster model $\mathfrak{C}$ of a complete theory T. Let A be a small subset of $\mathfrak{C}$.

Problem 1. Let L be the language which consists of binary relational symbols E_i, $i < \omega$. Let T be the theory in L saying that each E_i is an equivalence relation with infinitely many infinite classes.
(i) Let T_1 be the union of T and axioms saying that $E_0 \supseteq E_1 \supseteq \dots$ and each E_i-class splits into infinitely many E_{i+1}-classes. Prove that T_1 is complete, has q.e., is stable but not superstable, and is strongly dependent.
(ii) Let T_2 be the union of T and axioms saying that the relations $E_0, E_1, \dots$ are independent, i.e. the intersection of any classes of the relations $E_0, \dots, E_{n-1}$ is nonempty (for every n). Prove that T_2 is complete, has q.e., is stable but not strongly dependent.

Problem 2. (i) Prove that the theory of any linear order is dp-minimal.
(ii) Prove that each o-minimal theory is dp-minimal.

Problem 3.
(i) Prove that $\mathrm{Autf}_L(\mathfrak{C}/A) = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma(\bar{m}) \equiv_A^{Ls} \bar{m}\}$, where $\bar{m}$ is an enumeration of some $M \prec \mathfrak{C}$ containing A.
(ii) Prove that $\mathrm{Autf}_L(\mathfrak{C}/A)$ is the collection of all $\sigma \in \mathrm{Aut}(\mathfrak{C})$ which fix as sets all $\equiv_A^{Ls}$-classes on all (possibly infinite) products of sorts of $\mathfrak{C}$.
(iii) Prove that $\mathrm{Autf}_L(\mathfrak{C}/A)$ is the pointwise stabilizer in $\mathrm{Aut}(\mathfrak{C}/A)$ of the collection of all $\equiv_A^{Ls}$-classes on all (possibly infinite) products of sorts of $\mathfrak{C}$.

Problem 4. Prove that a global type p is Lstp_A-invariant if and only if for every A-indiscernible sequence $(a_i)_{i\in\omega}$ and $d \models p$, the sequence $(a_i)_{i\in\omega}$ is also Ad-indiscernible.

Problem 5. Check that $\mathrm{Autf}_L(\mathfrak{C}/A) \leq \mathrm{Autf}_{KP}(\mathfrak{C}/A) \leq \mathrm{Aut}(\mathfrak{C}/A)$ and that $\mathrm{Autf}_L(\mathfrak{C}/A)$ and $\mathrm{Autf}_{KP}(\mathfrak{C}/A)$ are normal subgroups of $\mathrm{Aut}(\mathfrak{C}/A)$.

Problem 6. Let $E(x, y)$ be an A-type-definable equivalence relation on some (small) product of sorts of $\mathfrak{C}$. Prove that $E(x, y) = \bigwedge_i E_i(x_i, y_i)$ for some A-type-definable equivalence relations E_i, where x_i and y_i are some corresponding countable subtuples of x and y, respectively.

Problem 7. Let $(a_i)_{i\in I}$ be representatives of all $\equiv_A^{KP}$-classes on countable products of sorts. They form a small subset of $\mathfrak{C}$. Let E be the equivalence relation (on a given product of sorts) defined by: $E(a, b)$ if and only if there exist $(a_i')_{i\in I}$ such that $(a_i')_{i\in I}a \equiv_A (a_i)_{i\in I}b$ and $a_i' \equiv_A^{KP} a_i$ for all $i \in I$. Let a, b be tuples of arbitrary (small) length. Prove that:
(i) E is the orbit equivalence relation of $\mathrm{Autf}_{KP}(\mathfrak{C}/A)$,
(ii) $\equiv_A^{KP} = E$,
(iii) $a \equiv_A^{KP} b$ if and only if $a_0 \equiv_A^{KP} b_0$ for every corresponding finite subtuples a_0 and b_0 of a and b, respectively.

Problem 8.
(i) Prove that $\mathrm{Autf}_{KP}(\mathfrak{C}/A) = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma(\bar{m}) \equiv_A^{KP} \bar{m}\}$, where $\bar{m}$ is an enumeration of some $M \prec \mathfrak{C}$ containing A.
(ii) Prove that $\mathrm{Autf}_{KP}(\mathfrak{C}/A)$ is the collection of all $\sigma \in \mathrm{Aut}(\mathfrak{C})$ which fix as sets all $\equiv_A^{KP}$-classes on all (possibly uncountable) products of sorts of $\mathfrak{C}$.

Problem 9. Let p be a type over A and $R(x, y)$ a bounded, A-invariant equivalence relation on $p(\mathfrak{C})$. Prove that:
(i) $\equiv_A^{Ls} \restriction_{p(\mathfrak{C})} \subseteq R$,
(ii) if R is type-definable, then $\equiv_A^{KP} \restriction_{p(\mathfrak{C})} \subseteq R$.
Comment. Try to follow the hint on page 40 of the notes.